Daily Quiz

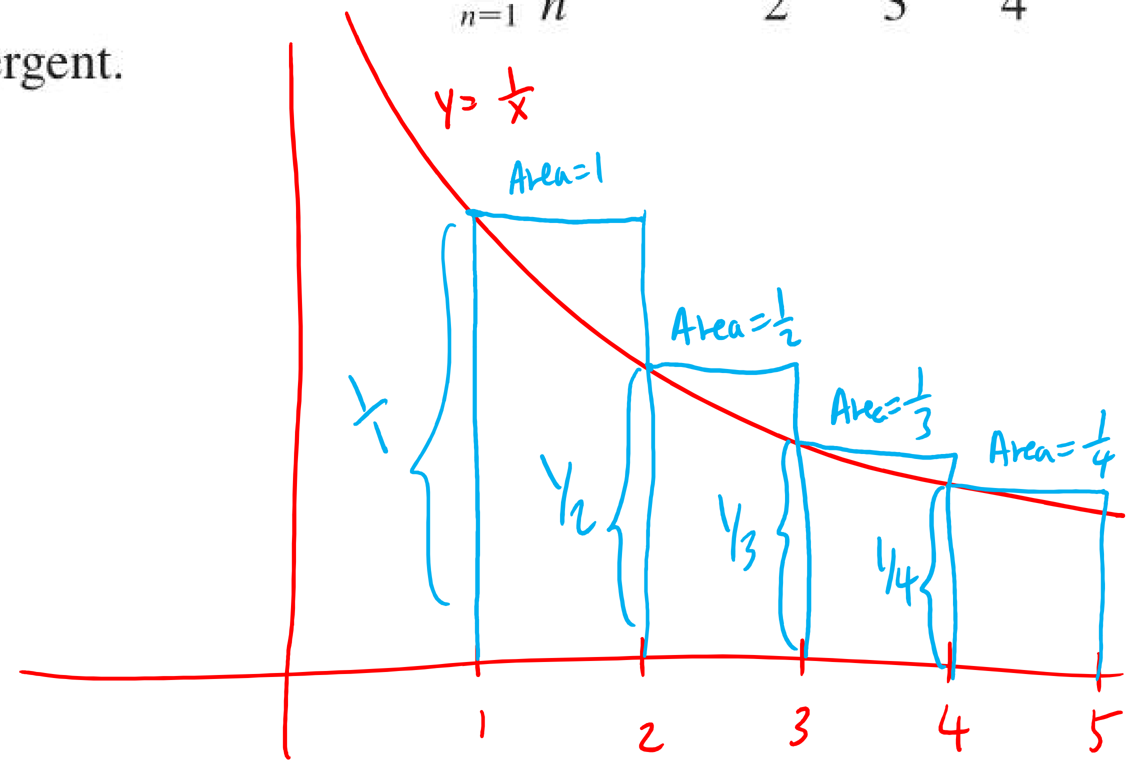
- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

8.2 Harmonic Series

V **EXAMPLE 7** Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.



Approximate $\int_1^{\infty} \frac{1}{x}\,dx$ using Left-Riemann sum with $\Delta x = 1$

Blue area > Red area

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots >$ Red area $= \int_1^{\infty} \frac{1}{x}\,dx = \infty$ from p-test

So the harmonic series diverges to ∞.

8.3 Integral Test

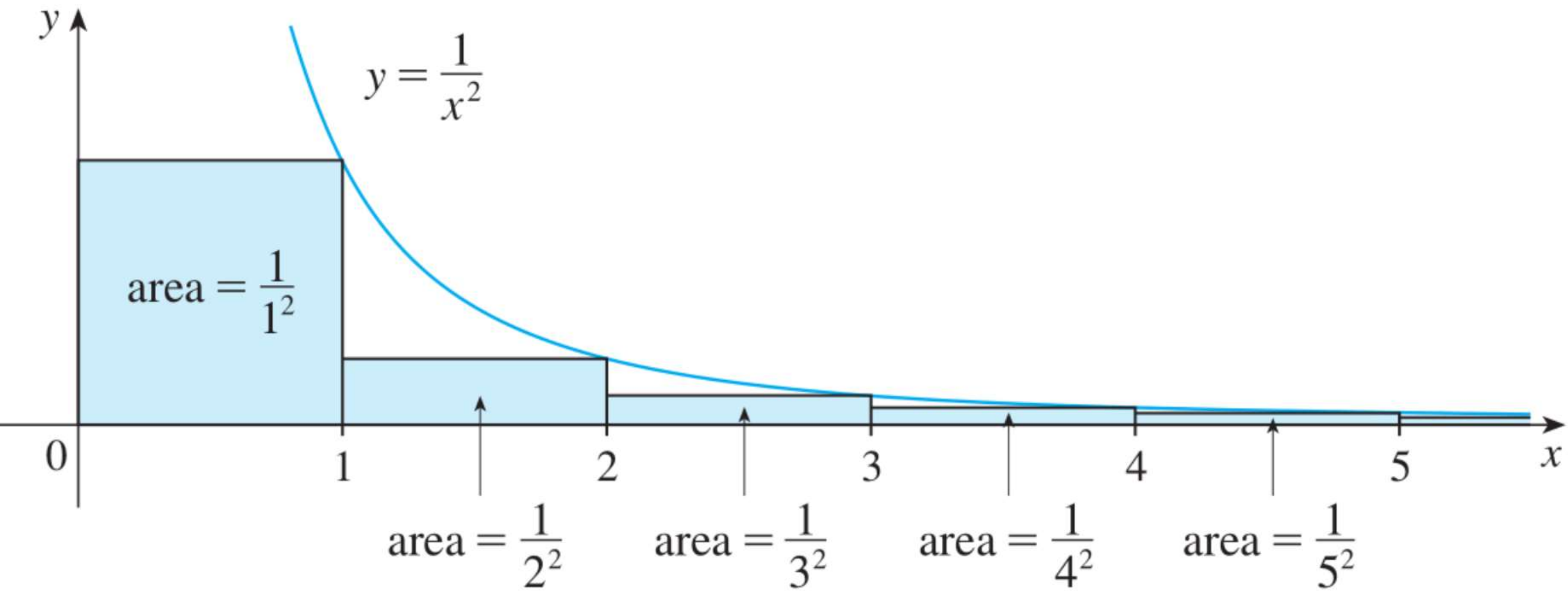
The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

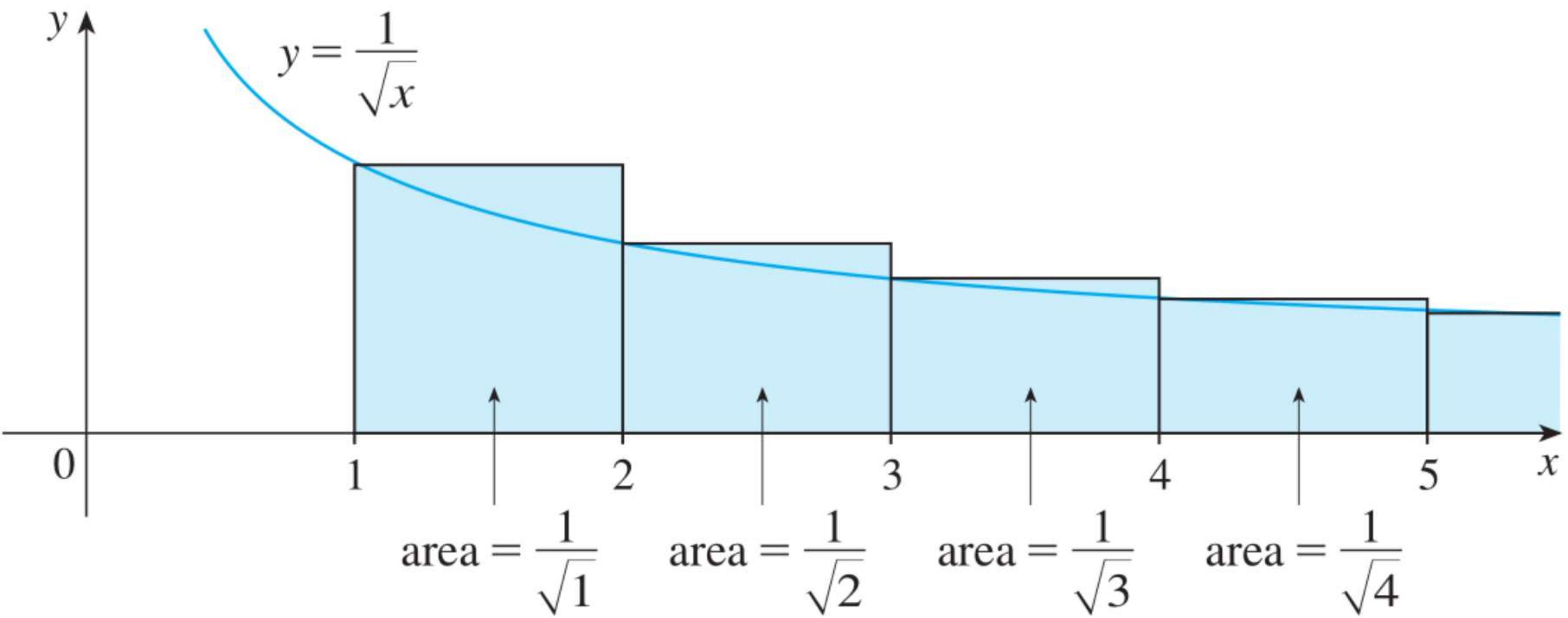
8.3 Convergent Series using the Integral Test

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \quad \text{versus} \quad \int_1^{\infty} \frac{1}{x^2}\, dx$$



8.3 Divergent Series using the Integral Test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots \text{ versus } \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$$



8.3 p-series

V EXAMPLE 2 Convergence of the p-series

For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

① $p < 0$

Ex: $\frac{1}{n^{-1}} = n$ so $\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} n^{-p}$ diverges

Using the divergence test

$$\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{-p}$$

but since $p<0$, $-p>0$

$$\lim_{n\to\infty} n^{-p} = \infty$$

so by the divergence test our series diverges.

② $p=0$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1 = \infty$$

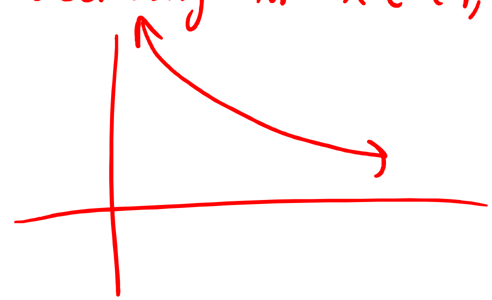
diverges.

③ $0 < p$

Use the integral test.

Check the hypotheses for $f(x) = \frac{1}{x^p}$.

$f(x)$ is continuous, positive, and decreasing for $x \in (1, \infty)$



8.3 p-series

V EXAMPLE 2 Convergence of the p-series

For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

By recalling the p-test for integrals, $\int_1^{\infty} \frac{1}{x^p}\,dx$ is diverging for $0 < p \le 1$ and is converging for $p > 1$.

Hence for $0 < p$, $p = 0$, and $0 < p \le 1$, our p-series diverges

while for $p > 1$, the p-series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{diverges for } p \le 1 \\ \text{converges for } p > 1 \end{cases}$$

8.3 p-test

The p-test for **integrals**:

$\boxed{2}$ $$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leqslant 1.$$

The p-test for **series**:

$\boxed{1}$ The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

8.3 Integral Test

V EXAMPLE 1 Using the Integral Test

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

$\int_1^{\infty} \frac{\ln x}{x}\, dx$ looks like it can be integrated with $u = \ln x$.

use the Integral Test

① check the hypotheses (continuity, positivity, and decreasing)

$f(x) = \frac{\ln x}{x} \quad x \in [1, \infty)$

$f(x)$ is positive for $x > 1$.

Find the derivative

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Since differentiability implies continuity, $f(x)$ is continuous. Observe that

$1 - \ln(x) < 0$ holds true for $x > e$

Since $\ln(e) = 1$. This shows that

$f'(x) < 0$ for $x > e$.

Therefore the function $f(x)$ satisfies all the hypotheses of the integral test, but only for x in (e, ∞).

8.3 Integral Test

V EXAMPLE 1 Using the Integral Test

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Let's compute the integral.

$$\int_e^{\infty} \frac{\ln x}{x}\,dx \qquad u = \ln x \qquad du = \frac{dx}{x}$$

$$= \int_1^{\infty} u\,du = \lim_{t\to\infty} \int_1^{t} u\,du$$

$$= \lim_{t\to\infty}\left[\frac{t^2}{2} - \frac{1}{2}\right] = \infty.$$

Since the integral diverges, by the Integral Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ also diverges.

8.3 Integral Test

Determine whether the series is convergent or divergent.

$$\int_1^{\infty} xe^{-x}\,dx \text{ looks integrable.}$$

$$\sum_{n=1}^{\infty} ne^{-n}$$

Let's use the integral test. Let $f(x) = xe^{-x}$. Since $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$, f is continuous on the interval $(1, \infty)$. Also, $f' < 0$ for $x > 1$ so f is decreasing on the interval $(1, \infty)$. Lastly, $f(x) = \frac{x}{e^x}$ is non-negative for x in $(1, \infty)$ so we satisfied the hypotheses for the integral test.

Computing the integral,

$u = x \quad dv = e^{-x}dx$

$du = dx \quad V = e^{-x}$

$$\int_1^\infty x e^{-x}\,dx = -xe^{-x}\Big]_1^\infty + \int_1^\infty e^{-x}\,dx$$

$$= \underbrace{\lim_{t\to\infty} -te^{-t}}_{=0 \text{ by L'Hospital}} + e^{-1} + \lim_{t\to\infty}\left[-e^{-x}\right]_1^t$$

$$= 0 + e^{-1} + \lim_{t\to\infty} -e^{-t} + e^{-1}$$

$$= e^{-1} + 0 + e^{-1}$$

$$= 2e^{-1}$$

$$= \frac{2}{e}.$$

Since the integral converges, the series $\sum_{n=1}^{\infty} ne^{-n}$ also converges by the integral test.

8.3 Integral Test

Note: When we use the Integral Test it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_4^{\infty} \frac{1}{(x-3)^2}\,dx$$

Also, it is not necessary that f be always decreasing. What is important is that f be *ultimately* decreasing, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^{\infty} a_n$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is convergent by Note 4 of Section 8.2.