Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Express $\dfrac{1}{(1+2x)^2}$ as a power series by differentiation.

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-2)^n x^n$$

$$\frac{d}{dx}\left(\frac{1}{1+2x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-2)^n x^n\right)$$

$$\frac{d}{dx}\left((1+2x)^{-1}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left((-2)^n x^n\right)$$

$$-1 \cdot (1+2x)^{-2} \cdot 2 = \sum_{n=0}^{\infty} (-2)^n \frac{d}{dx}(x^n)$$

$$\frac{-2}{(1+2x)^2} = \sum_{n=0}^{\infty} (-2)^n n x^{n-1}$$

$$\frac{1}{(1+2x)^2} = \sum_{n=0}^{\infty} (-2)^{n-1} n x^{n-1} = \boxed{\sum_{n=1}^{\infty} (-2)^{n-1} n x^{n-1}}$$

(a) Evaluate $\int \frac{1}{1+x^7}dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5} \frac{1}{1+x^7}dx$ correct to within 10^{-7}.

(a) $\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty}(-x^7)^n = \sum_{n=0}^{\infty}(-1)^n x^{7n}$; $\int \frac{1}{1+x^7}dx = \int \sum_{n=0}^{\infty}(-1)^n x^{7n}dx = \sum_{n=0}^{\infty}\int (-1)^n x^{7n}dx$

$= \sum_{n=0}^{\infty}(-1)^n \int x^{7n}dx = \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{7n+1}}{7n+1}\right) + C$

(b) $\int_0^{0.5}\frac{1}{1+x^7}dx = \sum_{n=0}^{\infty}(-1)^n\left[\frac{x^{7n+1}}{7n+1}\right]_0^{0.5} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{(0.5)^{7n+1}}{7n+1} - \frac{0}{7n+1}\right]$

$= \sum_{n=0}^{\infty}(-1)^n\frac{(0.5)^{7n+1}}{7n+1}$

Our goal is to approximate the above series using techniques that we learned, including the Alternating Series Estimation Theorem and the Remainder Estimate for the Integral Test.

Since we have an alternating series, let's use the Alternating Series Estimation Theorem.

To use the ASET, we need to first verify that $\sum_{n=0}^{\infty} (-1)^n \frac{(0.5)^{7n+1}}{7n+1}$ converges.

using the Ratio Test,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(0.5)^{7(n+1)+1}}{7(n+1)+1} \cdot \frac{7n+1}{(-1)^n(0.5)^{7n+1}} \right| = \lim_{n\to\infty} \frac{7n+1}{7n+8} \, \frac{(0.5)^{7n+8}}{(0.5)^{7n+1}}$$

$= (0.5)^7 = \left(\frac{1}{2}\right)^7 = \frac{1}{128}$. Since $L = \frac{1}{128} < 1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{(0.5)^{7n+1}}{7n+1}$ converges absolutely.

Now the ASET gives

$$|s - s_n| = |R_n| < b_{n+1} \leq 10^{-7}$$

$$|s - s_n| < \frac{(0.5)^{7(n+1)+1}}{7(n+1)+1} \leq 10^{-7}$$

$$\frac{(0.5)^{7n+8}}{7n+8} \le 10^{-7}.$$ To solve the inequality, we try a few values of n.

$n=1$ $$\frac{(0.5)^{7+8}}{7+8} = 2.03 \times 10^{-6}$$

$n=2$ $$\frac{(0.5)^{14+8}}{14+8} = 1.08 \times 10^{-8}$$

This shows that $n=2$ is the first integer that satisfies $b_{n+1} \le 10^{-7}$.

Therefore n needs to be greater than or equal to 2 in order for the partial sum $S_n = \sum_{k=0}^{n} (-1)^k \frac{(0.5)^{7k+1}}{7k+1}$ to be correct within 10^{-7}.

Since we need to compute the approximation,

$$\int_0^{0.5} \frac{1}{1+x^7}dx \approx S_2 = \sum_{n=0}^{2} (-1)^n \frac{(0.5)^{7n+1}}{7n+1} = 0.5 - \frac{(0.5)^8}{8} + \frac{(0.5)^{15}}{15} = 0.499514$$

8.7 Taylor Series Centered at 0 (Maclaurin Series)

The Taylor series of $f(x)$ centered at 0 is sometimes called the **Maclaurin Series.**

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

8.7 Maclaurin Series

Find the Maclaurin series of $f(x) = e^x$ and its interval of convergence.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

Compute the derivatives

$f = e^x \qquad f(0) = e^0 = 1$

$f' = e^x \qquad f'(0) = e^0 = 1$

$f'' = e^x \qquad f''(0) = e^0 = 1$

$f''' = e^x \qquad f'''(0) = e^0 = 1$

$\vdots \qquad \vdots$

$$T(x) = 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots$$

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Previously we computed the Maclaurin series of e^x.

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let's compute the interval of convergence using the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Since $L = 0 < 1$ for all values of x, the series $T(x)$ converges for all x

so the interval of convergence is $(-\infty, \infty)$.

1. Find the 4th degree Taylor polynomial for e^x centered at 0.

2. Use $T_4(x)$ to estimate e^2.

① $T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!}x^n = \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

$f(x) = e^x \quad f(0) = e^0 = 1$

$f'(x) = e^x \quad f'(0) = e^0 = 1$

$f''(x) = e^x \quad f''(0) = e^0 = 1$

$f'''(x) = e^x \quad f'''(0) = e^0 = 1$

$f^{(4)}(x) = e^x \quad f^{(4)}(0) = e^0 = 1$

$$T_4(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

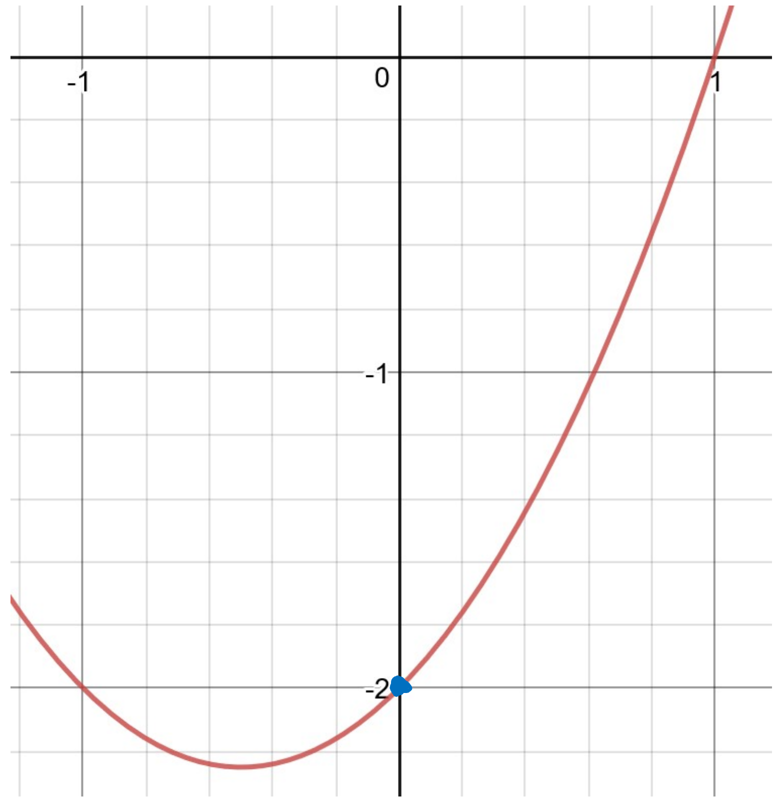
$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

② Since $e^x \approx T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$,

$$e^2 \approx T_4(2) = 1 + 2 + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} = 7$$

Note that $e^2 = 7.38906$

Suppose a function f has the following graph. If the 2nd degree Taylor polynomial centered at 0 for f is $T_2(x) = ax^2 + bx + c$, determine the signs of a, b, and c.



$$T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(0)}{n!} x^n = \frac{f^{(0)}(0)}{0!} x^0 + \frac{f^{(1)}(0)}{1!} x^1 + \frac{f^{(2)}(0)}{2!} x^2$$

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2} x^2$$

$$ax^2 + bx + c = \frac{f''(0)}{2} x^2 + f'(0)x + f(0)$$

$$a = \frac{f''(0)}{2}, \quad b = f'(0), \quad c = f(0).$$

From the graph, we see that $f(0)$ is negative, $f'(0)$ is positive (slope of f at 0 is positive), and $f''(0)$ is positive (concavity of f at 0 is positive).

Hence $a > 0$, $b > 0$, $c < 0$.

8.7 Taylor Series

Find the Taylor series for $f(x) = \cos x$ centered at 0.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Remember that the Taylor series encodes all the derivatives of a function at a point. If you know the derivatives, you determine the Taylor series.

$f = \cos x \qquad f^{(4)} = \cos x \qquad f^{(8)} = \cos x$

$f' = -\sin x \qquad f^{(5)} = -\sin x \qquad \vdots$

$f'' = -\cos x \qquad f^{(6)} = -\cos x \qquad \vdots$

$f''' = \sin x \qquad f^{(7)} = \sin x$

$f(0) = \cos 0 = 1 \qquad f^{(4)}(0) = \cos 0 = 1$

$f'(0) = -\sin 0 = 0 \qquad f^{(5)}(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1 \qquad f^{(6)}(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0 \qquad f^{(7)}(0) = \sin 0 = 0$

$\vdots$

$$T(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \frac{-1}{6!}x^6 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$$

(even powers survived!)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

8.7 Taylor Series

Find the Taylor series for $f(x) = \sin x$ centered at 0.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Compute the derivatives

$f = \sin x$ $\quad f^{(4)} = \sin x$ $\quad f^{(8)} = \sin x$

$f' = \cos x$ $\quad f^{(5)} = \cos x$ $\quad \vdots$

$f'' = -\sin x$ $\quad f^{(6)} = -\sin x$

$f''' = -\cos x$ $\quad f^{(7)} = -\cos x$

$f(0) = \sin 0 = 0$ $\quad f^{(4)}(0) = \sin 0 = 0$

$f'(0) = \cos 0 = 1$ $\quad f^{(5)}(0) = \cos 0 = 1$

$f''(0) = -\sin 0 = 0$ $\quad f^{(6)}(0) = -\sin 0 = 0$

$f'''(0) = -\cos 0 = -1$ $\quad f^{(7)}(0) = -\cos 0 = -1$

$\vdots$

$$T(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$ (odd powers survived!)

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

List of power series (centered at 0) that you must memorize. "I" means Interval of Convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad \text{I: } (-1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{I: } (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{I: } (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{I: } (-\infty, \infty)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{I: } [-1, 1]$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{I: } (-1, 1]$$