Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

8.5 Power Series

Find the radius of convergence and the interval of convergence of the following series.

$$\sum_{n=1}^{\infty} \frac{(8x+3)^n}{n^2}$$

use the Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(8x+3)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(8x+3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(8x+3)^{n+1}}{(8x+3)^n} \right| \cdot \frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} |8x+3| \cdot \frac{n^2}{(n+1)^2}$$

$$= |8x+3| \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= |8x+3| \cdot 1 = |8x+3|$$

By the Ratio Test, the series converges absolutely when $L = |8x+3| < 1$.

To find the radius of convergence, we need to normalize the coefficient of x.

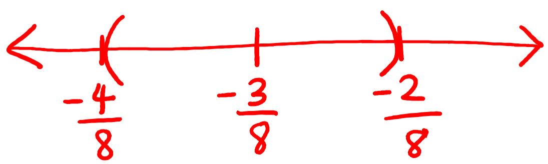
$$\frac{1}{8}|8x+3| < \frac{1}{8} \cdot 1$$

$$\left| \frac{8x+3}{8} \right| < \frac{1}{8}$$

$$\left| x + \frac{3}{8} \right| < \frac{1}{8}$$

Hence the radius of convergence is $\frac{1}{8}$ and the interval of convergence is centered at $-\frac{3}{8}$.

Possible interval of convergence



Check the boundary points

① $x = -\frac{4}{8}$

$$\sum_{n=1}^{\infty} \frac{\left(8\left(-\frac{4}{8}\right)+3\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Since $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test $(p=2)$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

② $x = -\frac{2}{8}$

$$\sum_{n=1}^{\infty} \frac{\left(8\left(-\frac{2}{8}\right)+3\right)}{n^2} = \sum_{n=1}^{\infty} \frac{1^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test $(p=2)$.

Hence both $x = -\frac{4}{8}$ and $x = -\frac{2}{8}$ are included in the interval of convergence.

Therefore the interval of convergence is

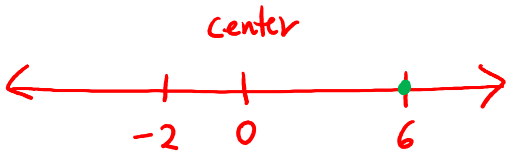
$$\left[-\frac{4}{8}, -\frac{2}{8}\right]$$ and the radius of convergence is $R = \frac{1}{8}$.

8.5 Power Series

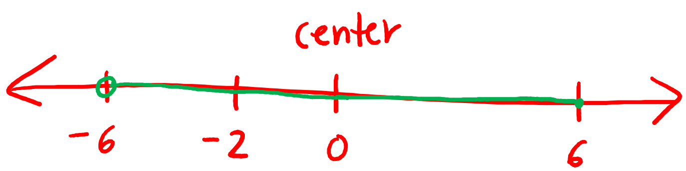
Determine whether the statement is true or false.

If $\sum c_n 6^n$ is convergent, then $\sum c_n(-2)^n$ is convergent.

Suppose we have a power series $\sum c_n x^n$. The power series is centered at 0.
Since $\sum c_n 6^n$ converges, $\sum c_n x^n$ converges at $x=6$. On the number line, we have



Note that the distance from the center to a known point of convergence is 6, which means that the radius of convergence is at least 6. Since -2 is inside the radius, we conclude that $\sum c_n x^n$ converges at $x=-2$ and so $\sum c_n(-2)^n$ converges.



Note: We don't know what happens on the boundary $x=-6$.

Find a power series representation for $f(x) = \dfrac{1}{1-x}$.

Recall: $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r| < 1$.

Let $a = 1$ and $r = x$. Then for $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

So the power series representation for $f(x) = \dfrac{1}{1-x}$ is $\displaystyle\sum_{n=0}^{\infty} x^n$.

A function $f(x)$ is equal to its power series only for x in the interval of convergence.

Example: Let $f(x) = \dfrac{1}{1-x}$. Then

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

only for x in the interval of convergence $(-1, 1)$. If x is **not in the interval of convergence**, then the function is not equal to its power series at x:

$$\frac{1}{1-x} \neq \sum_{n=0}^{\infty} x^n$$

https://www.desmos.com/calculator/u0jgbq7jtj

8.6 Things you can do with power series

Substitution: Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ and let u be a polynomial in x.

Then

$$f(u) = \sum_{n=0}^{\infty} c_n (u-a)^n$$

Express $\dfrac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Ratio Test

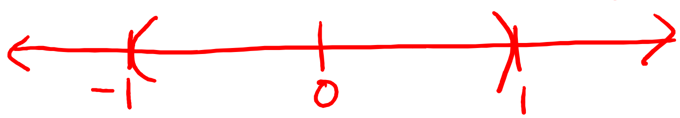
$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{2(n+1)}}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty} |x^2| = |x|^2.$$

By the Ratio Test, the power series converges when $|x|^2 < 1$.

Equivalently, $|x| < 1$.

Possible interval of convergence



Check the boundary points $x=-1$, $x=1$ for convergence.

$x=-1$ $\quad \displaystyle\sum_{n=0}^{\infty} (-1)^n (-1)^{2n} = \sum_{n=0}^{\infty} (-1)^n 1^n = \sum_{n=0}^{\infty} (-1)^n$

$x=1$ $\quad \displaystyle\sum_{n=0}^{\infty} (-1)^n 1^{2n} = \sum_{n=0}^{\infty} (-1)^n$

Since $\lim\limits_{n\to\infty} |(-1)^n| = 1 \neq 0$, both series diverge by the Divergence Test.

Hence the interval of convergence is

$$(-1, 1).$$

Find a power series representation for $\dfrac{1}{x+2}$.

$$\frac{1}{x+2} = \frac{\frac{1}{2}}{\frac{1}{2}(x+2)} = \frac{\frac{1}{2}}{1+\frac{x}{2}} = \frac{1}{2}\cdot\frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$

8.6 Things you can do with power series

Multiply by a polynomial in x: Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and let $p(x)$ be a polynomial in x. Then

$$p(x) \cdot f(x) = \sum_{n=0}^{\infty} p(x) \cdot c_n(x-a)^n$$

Note: Division is okay as long as there aren't any negative powers of x left over after simplification.

$$\frac{x^2 + x^3 + x^4 + \cdots}{x^2} = 1 + x + x^2 + \cdots \quad \text{POWER SERIES}$$

$$\frac{x^2 + x^3 + x^4 + \cdots}{x^3} = \frac{1}{x} + 1 + x + \cdots \quad \text{NOT A POWER SERIES}$$

Find a power series representation for $\dfrac{x^3}{x+2}$.

$$\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^3 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{2^{n+1}}$$

8.6 Things you can do with power series

Term-by-term Differentiation (Swapping the sum and the differential operator):

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$$

Express $\dfrac{1}{(1-x)^2}$ as a power series by differentiation. What is its interval of convergence?

Observe that $\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)$.

Then $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

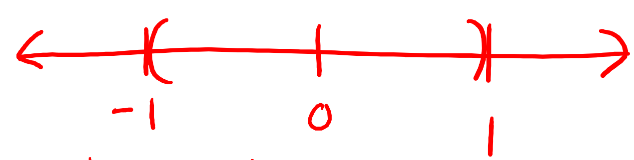
$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$$

Use the Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,x^n}{n\,x^{n-1}}\right| = \lim_{n\to\infty}|x|\,\frac{n+1}{n} = |x|$$

The series converges absolutely when $|x|<1$.

Possible interval of convergence



Check the boundary points $x=-1$, $x=1$ for convergence

$x=-1$: $\sum_{n=1}^{\infty} n(-1)^{n-1}$

$x=1$: $\sum_{n=1}^{\infty} n\,1^{n-1} = \sum_{n=1}^{\infty} n$

Since $\lim_{n\to\infty}\left|n(-1)^{n-1}\right| = \infty$ and $\lim_{n\to\infty} n = \infty$

both series diverge by the Divergence Test.

Therefore the interval of converge is $(-1,1)$.

Express $\dfrac{x}{(1+2x)^3}$ as a power series by differentiation.

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-2)^n x^n$$

$$\frac{d}{dx}\left(\frac{1}{1+2x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-2)^n x^n\right) = \sum_{n=0}^{\infty} (-2)^n \frac{d}{dx}(x^n)$$

$$\frac{-1}{(1+2x)^2} \cdot 2 = \sum_{n=0}^{\infty} (-2)^n n x^{n-1} = \sum_{n=1}^{\infty} (-2)^n n x^{n-1}$$

$$\frac{d}{dx}\left(\frac{-2}{(1+2x)^2}\right) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} (-2)^n n x^{n-1}\right) = \sum_{n=1}^{\infty} (-2)^n n \frac{d}{dx}(x^{n-1})$$

$$\frac{4}{(1+2x)^3} \cdot 2 = \sum_{n=1}^{\infty} (-2)^n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} (-2)^n n(n-1) x^{n-2}$$

$$\frac{8}{(1+2x)^3} = \sum_{n=2}^{\infty} (-2)^n n(n-1) x^{n-2}$$

$$\frac{1}{(1+2x)^3} = \sum_{n=2}^{\infty} \frac{(-2)^n}{8} n(n-1) x^{n-2}$$

$$\boxed{\frac{x}{(1+2x)^3} = \sum_{n=2}^{\infty} \frac{(-2)^n}{8} n(n-1) x^{n-1}}$$

8.6 Things you can do with power series

Term-by-term Integration (Swapping the sum and the integral):

$$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n \right] dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx$$

Find a power series representation for $\ln(1+x)$ and its interval of convergence.

Observe that $\int_0^x \frac{1}{1+t}\,dt = \ln(1+x)$

Then $\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \frac{1}{1-(-t)}\,dt$

$$= \int_0^x \sum_{n=0}^{\infty} (-t)^n\,dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt = \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{n+1}}{n+1}\right]_0^x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

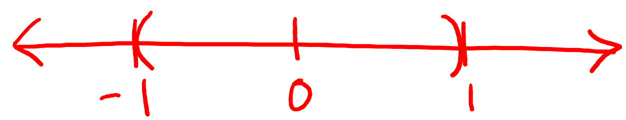
Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{(n+1)+1}}{(n+1)+1} \cdot \frac{n+1}{(-1)^n x^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n+2} |x| = |x|$$

By the Ratio Test the power series converges absolutely when $L = |x| < 1$.

Possible Interval of Convergence



check the boundary points $x=-1$, $x=1$ for convergence.

$x=-1$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-1}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1} = -\sum_{n=1}^{\infty} \frac{1}{n}$$ diverges by p-test

$x=1$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$ converges by alternating series test since

① $\lim_{n\to\infty} \frac{1}{n} = 0$ and

② $\frac{1}{n} \geq \frac{1}{n+1}$ for all $n \geq 1$

Therefore the interval of convergence is $(-1, 1]$.

Find a power series representation for $\arctan x$ and its interval of convergence.

Observe that $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$.

Then since $\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}$,

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

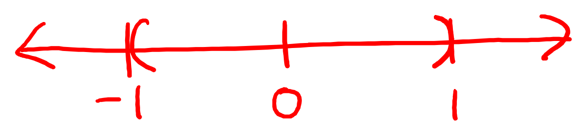
Use Ratio Test to find the interval of convergence.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n x^{2(n+1)+1}}{2(n+1)+1} \cdot \frac{2n+1}{(-1)^n x^{2n+1}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{2n+1}{2n+3}\right| = |x|^2$$

By the Ratio Test, the power series converges when $L = |x|^2 < 1$.

Taking square roots on both sides, $|x| < 1$. Hence we have possible interval of convergence



Check the boundary points $x=-1$, $x=1$ for convergence.

$x=-1$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)}{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$x=1$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by Alternating Series Test since

① $\lim_{n\to\infty} \frac{1}{2n+1} = 0$

② $\frac{1}{2n+1} \geq \frac{1}{2(n+1)+1} = \frac{1}{2n+3}$

Therefore both boundary points are in the interval of convergence and

$[-1, 1]$ is the interval of convergence.