Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

8.2 What can we do with sequences?

Let $\{a_n\}$ be a sequence. What can we do with sequences?

1. Infinite sums called **Series**

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

2. Infinite products

$$a_1 \cdot a_2 \cdot a_3 \cdot \cdots = \prod_{n=1}^{\infty} a_n$$

8.2 Review of Sigma Notation

$$\sum_{k=1}^{n} a_k$$

- n: upper limit of summation "stopping index"
- $\sum$: summation sign
- a_k: k^{th} term
- k: index of summation
- 1: lower limit of summation "starting index"

8.2 Series

Definition. Given a sequence $\{a_k\}$, a finite sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

is called the n**-th partial sum** s_n.

Series is an infinite sum of the sequence $\{a_k\}$,

$$\sum_{k=1}^{\infty} a_k$$

8.2 Series

Since infinity is not a number, we define the series as the limit of the partial sums:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

If the limit exists and is equal to a real number s, we say that the series **converges** and write

$$s = \sum_{k=1}^{\infty} a_k.$$

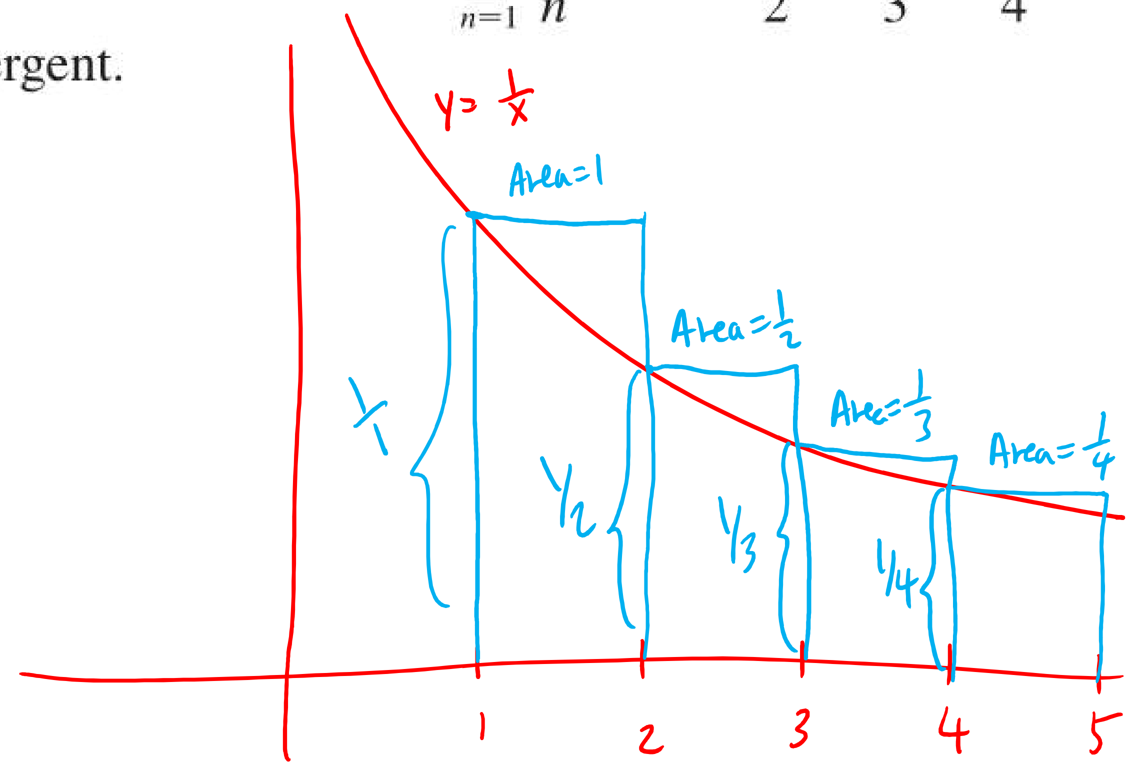
If the limit doesn't exist, then we say that the series is **divergent**.

8.2 Harmonic Series

EXAMPLE 7 Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.



Approximate $\int_1^\infty \frac{1}{x}\,dx$ using Left-Riemann sum with $\Delta x = 1$

Blue area > Red area

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots >$ Red area

$= \int_1^\infty \frac{1}{x}\,dx$

$= \infty$ from p-test

So the harmonic series diverges to ∞.

8.2 Harmonic Series

Another way to see the divergence of the **harmonic series** is to compare the expanded form to another series that diverges.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots \\ &> 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\cdots \\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots \\ &= \infty\end{aligned}$$

Therefore $\sum_{n=1}^{\infty}\frac{1}{n} \geq \infty$ and the harmonic series **diverges**.

8.2 Telescoping Sum

Find the partial sum s_4 and show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

$$s_4 = \sum_{n=1}^{4} \frac{1}{n(n+1)} = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \frac{1}{4(4+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Use partial fractions

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} + \frac{-1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

8.2 Telescoping Sum

Find the partial sum s_4 and show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

$$\sum_{n=1}^{5} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) = 1 - \frac{1}{6}$$

$$\sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) + \left(\frac{1}{m} - \frac{1}{m+1}\right)$$
$$= 1 - \frac{1}{m+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{m\to\infty} \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{m\to\infty} 1 - \frac{1}{m+1} = 1 - 0 = 1$$

8.2 The Divergence Test

Theorem. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

The Divergence Test.

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(WRONG) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Counterexample:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{but} \quad \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

8.2 The Divergence Test

Using the Test for Divergence Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

use divergence test

$$a_n = \frac{n^2}{5n^2+4}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$$

so by the Divergence Test,

the series diverges.

8.2 The Divergence Test

Note 1: A finite number of terms doesn't affect the convergence or divergence of a series. What matters is the **long-term** behavior, not the short term.

Note 2: The converse of Theorem 6 is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\Sigma\, a_n$ is convergent. Observe that for the harmonic series $\Sigma\, 1/n$ we have $a_n = 1/n \to 0$ as $n \to \infty$, but we showed in Example 7 that $\Sigma\, 1/n$ is divergent.

Note 3: If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\Sigma\, a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence of $\Sigma\, a_n$. Remember the warning in Note 2: If $\lim_{n\to\infty} a_n = 0$, the series $\Sigma\, a_n$ might converge or it might diverge.

Useful Limits

1. $\lim_{x\to\infty} \sin x = \text{ DNE}$

2. $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$

3. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

4. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

5. $\lim_{n\to\infty} \sqrt[n]{n} = 1$