Daily Quiz

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8.5 Power Series

EXAMPLE 4 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$(x-0)^n$ center

Use the Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{(n+1)+1}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-3)^{n+1}}{(-3)^n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$$

$$= \lim_{n\to\infty} \left| (-3) \cdot x \cdot \sqrt{\frac{n+1}{n+2}} \right|$$

$$= |-3x| \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}}$$

$$= 3|x| \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}}$$

by continuity of $\sqrt{\ }$

$$= 3|x| \sqrt{\lim_{n\to\infty} \frac{n+1}{n+2}}$$

$$= 3|x| \cdot 1$$

$$= 3|x|$$

Hence $L = 3|x|$.

By the Ratio Test, three things are true:

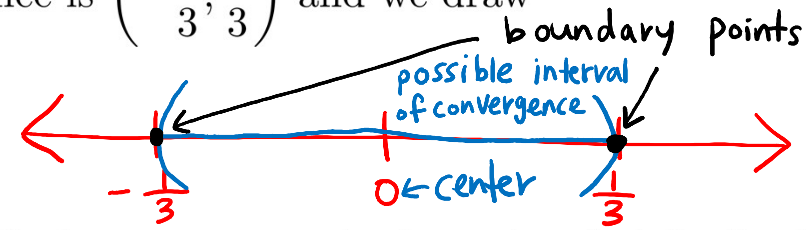
1. For values of x where $L = 3|x| < 1$, the power series converges absolutely.
2. For values of x where $L = 3|x| > 1$, the power series diverges.
3. For values of x where $L = 3|x| = 1$, the Ratio Test is inconclusive and we need to use other methods to determine convergence at these values of x.

Observe that the inequality $3|x| < 1$ contains the information about the Radius of Convergence. **When the coefficient of x is 1**, the number on the right-hand side becomes the **Radius of Convergence**:

$$3|x| < 1 \iff |x| < \frac{1}{3}$$

Therefore the **Radius of Convergence** is $R = \dfrac{1}{3}$.

To organize the convergence results from the Ratio Test, we draw a number line to indicate the **interval of convergence**: $|x| < \frac{1}{3}$ describes the values of x that are less than $\frac{1}{3}$ distance away from 0. Hence a possible interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$ and we draw



Observe that our power series is centered at 0. On the interval of convergence above, the distance from the center 0 to the boundary of the interval is $\frac{1}{3}$. In other words, the **Radius of Convergence** is the distance from the center to the boundary on the interval of convergence, and it is equal to $\frac{1}{3}$ for this example. We excluded $|x| > 1$ from the interval of convergence because the power series diverges at those values of x.

Note that the Ratio Test gives no information about the **boundary points** when $3|x| = 1$. The two boundary points $x = -\frac{1}{3}$ and $x = \frac{1}{3}$ must be checked manually using other convergence tests.

Check the boundary points

① When $x = -\frac{1}{3}$,

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{\left(-3 \cdot -\frac{1}{3}\right)^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

Compare $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ using the Limit Comparison Test.

Since $\frac{1}{\sqrt{n+1}}$ and $\frac{1}{\sqrt{n}}$ are positive for $n \geq 1$,

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \sqrt{1} = 1 > 0.$$

Therefore either both $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge or both diverge. But since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges via p-test $\left(p = \frac{1}{2}\right)$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ also diverges and so our power series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ diverges when $x = -\frac{1}{3}$.

② When $x = \frac{1}{3}$,

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{\left(\frac{-3}{3}\right)^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is not absolutely convergent since we've already checked from the previous step.

It is however conditionally convergent since by the Alternating Series Test,

① $\lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0$ (vanishing at infinity)

② $\frac{1}{\sqrt{n+1}} \geq \frac{1}{\sqrt{n+2}}$ (decreasing)

Hence the power series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ converges when $x = \frac{1}{3}$.

Therefore the interval of convergence is

$$\left(-\frac{1}{3}, \frac{1}{3}\right].$$

8.5 Power Series Centered at A

Definition. A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series centered at** a.

Ratio Test for Power Series Centered at a.

Given a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, let

$$L(x) = \lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|.$$

1. For values of x where $L(x) < 1$, the power series is absolutely convergent.
2. For values of x where $L(x) > 1$, the power series diverges.
3. For values of x where $L(x) = 1$, the Ratio Test is inconclusive and we must use other testing methods to determine convergence.

8.5 Radius of Convergence

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

1. The series converges only when $x = a$. $(R = 0)$
2. The series converges for all x. $(R = \infty)$
3. There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

Definition. The number R is called the **radius of convergence** of the power series.

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

Computing Interval of Convergence

Make sure the coefficient of x is 1.

$$|x-a|<R$$

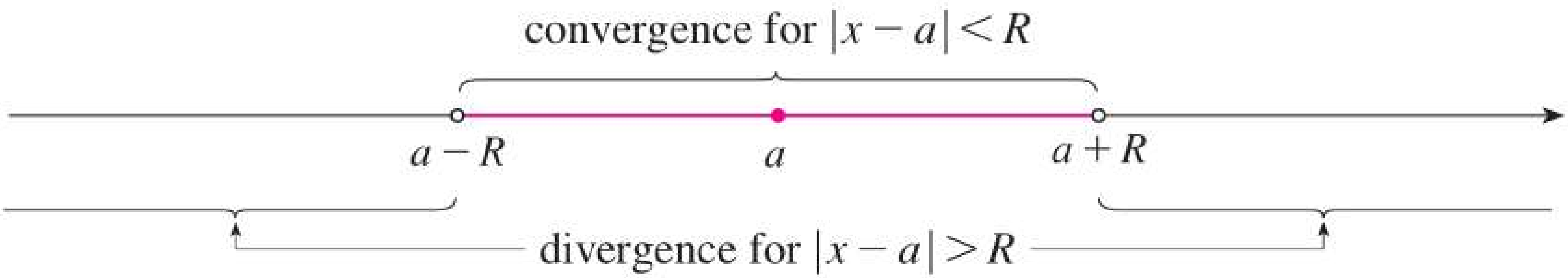
center of the power series

Radius of Convergence

Boundary points of the Interval of Convergence

WARNING: When x is a *boundary point* of the interval, that is, $x = a \pm R$, anything can happen - the series might converge at one or both boundary points or it might diverge at both boundary points. Thus when R is positive and finite, there are four possibilities for the interval of convergence:

$$(a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R), \quad [a-R, a+R]$$



8.5 Power Series

EXAMPLE 2 **Using the Ratio Test to determine where a power series converges**

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

center

Use the Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(x-3)^n}\cdot\frac{n}{n+1}\right|$$

$$= \lim_{n\to\infty}\left|(x-3)\cdot\frac{n}{n+1}\right| = \lim_{n\to\infty}|x-3|\cdot\left|\frac{n}{n+1}\right|$$

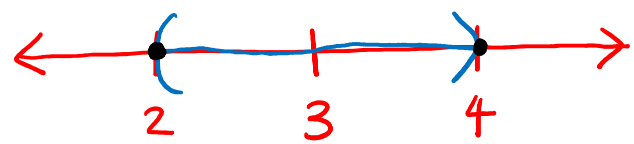
$$= |x-3|\lim_{n\to\infty}\frac{n}{n+1}$$

$$= |x-3|\cdot 1 = |x-3|$$

By Ratio Test, the series converges absolutely when $L = |x-3| < 1$.

Since the coefficient of x is 1, the radius of convergence is the right-hand side, $R=1$.

possible interval of convergence:



Inconclusive for boundary points $x=2$, $x=4$.

8.5 Power Series

EXAMPLE 2 **Using the Ratio Test to determine where a power series converges**

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Check Boundary points $x=2, 4$.

① When $x=2$,

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ (this is the alternating harmonic series).

Since the alternating harmonic series converges conditionally, our series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges at $x=2$.

② When $x=4$,

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$ (harmonic series)

Since the harmonic series diverges, our series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ diverges at $x=4$. Therefore, the interval of convergence is $[2, 4)$.

8.5 Power Series

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

8.5 Power Series

V EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$x-(-2)$ centered at $a=-2$.

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

Ratio Test to find the radius of convergence.

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)(x+2)^{n+1}}{3^{(n+1)+1}}\cdot\frac{3^{n+1}}{n(x+2)^n}\right|$$

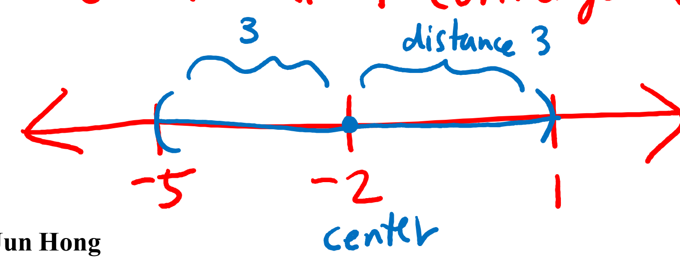
$$=\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{3^{n+1}}{3^{n+2}}\cdot\left|\frac{(x+2)^{n+1}}{(x+2)^n}\right|$$

$$=\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{1}{3}\cdot|x+2|$$

$$=\frac{|x+2|}{3}\lim_{n\to\infty}\frac{n+1}{n}=\frac{|x+2|}{3}\cdot 1=\frac{|x+2|}{3}$$

By the Ratio Test, the given series $\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$ converges absolutely for $L=\frac{|x+2|}{3}<1$. To find the radius of convergence, the coefficient of x must be 1. Hence $|x+2|<3$ and the radius of convergence is $R=3$ with the center $a=-2$.

Possible interval of convergence:



Check the boundary points $x=-5$ and $x=1$.

① When $x=-5$,

$$\sum_{n=0}^{\infty}\frac{n(-5+2)^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{n(-3)^n}{3^{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{n}{3}\left(\frac{-3}{3}\right)^n=\sum_{n=0}^{\infty}\frac{n}{3}\cdot(-1)^n$$

The series above is divergent since $\lim_{n\to\infty}\frac{n}{3}=\infty$. (Divergence Test)

Hence $\sum\frac{n(x+2)^n}{3^{n+1}}$ diverges when $x=-5$

② When $x=1$,

$$\sum_{n=0}^{\infty}\frac{n(1+2)^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{n}{3}\cdot\left(\frac{3^n}{3^n}\right)=\sum_{n=0}^{\infty}\frac{n}{3}$$

which diverges since $\lim_{n\to\infty}\frac{n}{3}=\infty$ (Divergence Test).

Therefore the interval of convergence for the power series $\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$ is

$$(-5,1)$$