Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Let $f(x) = 1 + x + x^2$. Find the 2nd degree ($k=2$) Taylor polynomial of $f(x)$ centered at 1. ($a=1$)

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1 + \frac{f^{(2)}(1)}{2!}(x-1)^2$$

$f(x) = 1 + x + x^2$ $\quad f(1) = 1 + 1 + 1^2 = 3$

$f'(x) = 1 + 2x$ $\quad f'(1) = 1 + 2\cdot 1 = 3$

$f''(x) = 2$ $\quad f''(1) = 2$

$$T_2(x) = \frac{3}{0!}(x-1)^0 + \frac{3}{1!}(x-1)^1 + \frac{2}{2!}(x-1)^2$$

$$\boxed{T_2(x) = 3 + 3(x-1) + (x-1)^2}$$

Observation: If we multiply everything out, we get

$$T_2(x) = 3 + 3(x-1) + (x-1)^2 = 3 + 3x - 3 + x^2 - 2x + 1$$
$$= (3-3+1) + (3x-2x) + x^2 = 1 + x + x^2.$$

Since $f(x)$ is a polynomial, $T_2(x)$ is the same polynomial but just rewritten with center $a=1$.

Approximating functions using polynomials is very useful!

Example: Approximating $f(x) = \cos x$ with Taylor polynomials centered at 0.

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + \frac{f^{(k)}(0)}{k!} x^k$$

https://www.desmos.com/calculator/chybqs87ex

Observation: We get better and better approximation as we increase the degree k of the Taylor polynomial $T_k(x)$.

What happens if we let $k \to \infty$? Can a function be **equal** to the limit of its Taylor polynomials?

In some cases, a function is indeed equal to its **Taylor series** $T(x)$. We define the **Taylor series of a function** $f(x)$ **centered at 0** as

$$\begin{aligned} T(x) &= \lim_{k\to\infty} T_k(x) \\ &= \lim_{k\to\infty} \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n \end{aligned}$$

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

To study Taylor series, we need to first understand the properties of a more general mathematical object called the **Power Series**.

8.5 Power Series

A **power series** (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

A power series may converge for some values of x and diverge for other values of x. Note that **the power series resembles a polynomial.** The only difference is that it has infinitely many terms.

8.5 Power Series

Example. The Taylor series centered at 0,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is a type of a power series where $c_n = \dfrac{f^{(n)}(0)}{n!}$.

Example. If we take $c_n = 1$ for all n, the power series becomes the **geometric series**

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

For what values of x is the geometric series convergent?

Use the Ratio Test:

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x^n$, $a_n = x^n$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x^n \cdot x}{x^n} \right| = \lim_{n\to\infty} |x|$

Since $|x|$ is independent of n,

$L = \lim_{n\to\infty} |x| = |x|$.

By the Ratio Test the series converges absolutely when

$L = |x| < 1$.

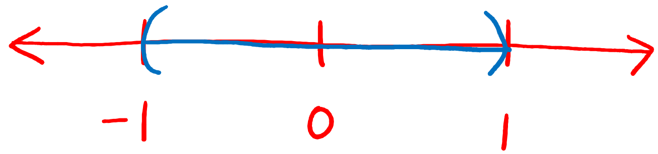
The series diverges when $L = |x| > 1$; inconclusive and further testing is needed when $L = |x| = 1$.

Inconclusive: $|x| = 1$, $x = 1, -1$.

Check $x = 1$: Plug in $x = 1$ to the series $\sum_{n=0}^{\infty} x^n$ to get $\sum_{n=0}^{\infty} 1^n$. This series diverges by the Divergence Test.

Check $x = -1$: Plug in $x = -1$ to the series $\sum_{n=0}^{\infty} x^n$ to get $\sum_{n=0}^{\infty} (-1)^n$. This series diverges by the Divergence Test.

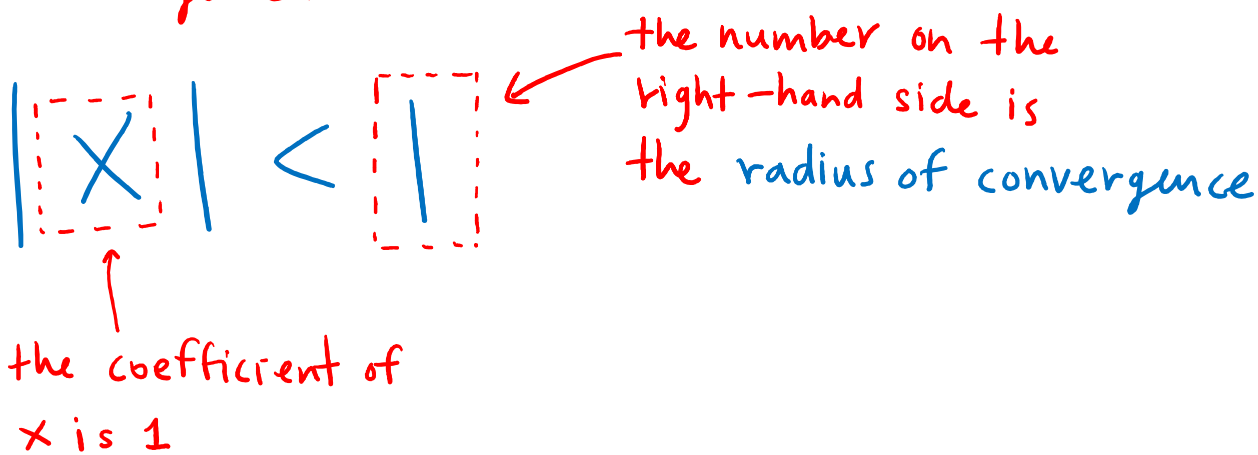
To organize the convergence results above, we draw a number line to indicate the interval of convergence. $|x|<1$ describes the values of x that are less than 1 distance away from 0. Hence the interval of convergence is $(-1,1)$ and we can draw



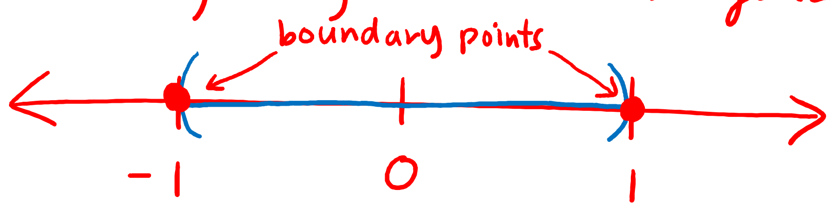
We excluded $x=1,-1$ and $|x|>1$ from the interval of convergence because the power series diverges at those values of x.

Observe that the power series $\sum_{n=0}^{\infty} x^n$ is centered at 0. On the interval of convergence above, the distance from the center 0 to the boundary of the interval is 1. In other words, the Radius of Convergence is the distance from the center to the boundary on the interval of convergence, and it is equal to 1 in this example.

Observe that the inequality $|x| < 1$ gives away the radius of convergence. When the coefficient of x is 1, the number on the right-hand side becomes the radius of convergence:



However, note that the radius of convergence gives no information about the boundary points of the interval of convergence. These two points must be checked manually using other convergence tests.



Ratio Test for Power Series Centered at 0.

Given a power series $\sum_{n=0}^{\infty} c_n x^n$, let $L(x) = \lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$.

1. For values of x where $L(x) < 1$, the power series is absolutely convergent.
2. For values of x where $L(x) > 1$, the power series diverges.
3. For values of x where $L(x) = 1$, the Ratio Test is inconclusive and we must use other testing methods to determine convergence.

8.5 Power Series

Theorem. For a given power series (centered at 0) $\sum_{n=0}^{\infty} c_n x^n$ there are only three possibilities:

1. The series converges only when $x = 0$. ($R = 0$)
2. The series converges for all x. ($R = \infty$)
3. There is a positive number R such that the series converges if $|x| < R$ and diverges if $|x| > R$.

Definition. The number R is called the **radius of convergence** of the power series.

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

8.5 Power Series – Finding the Radius of Convergence

The Ratio Test is used to determine the radius of convergence R.

The Ratio Test is inconclusive on the boundary of the interval of convergence, so the **endpoints must be checked with other tests** such as the divergence test, integral test, direct comparison test, limit comparison test, or the alternating series test.

8.5 Power Series

Remark: The power series is centered at 0.

EXAMPLE 1 **A power series that converges only at its center**

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

We use the Ratio Test

Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\ x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cdot n!}{n!} \ \frac{x^n \cdot x}{x^n} \right|$$

$$L = \lim_{n\to\infty} \left| (n+1)\cdot x \right| = \lim_{n\to\infty} |n+1| \cdot |x|$$

For $x \neq 0$,

$$L = \lim_{n\to\infty} |n+1| \cdot |x| = |x| \cdot \lim_{n\to\infty} (n+1) = |x| \cdot \infty = \infty$$

For $x = 0$,

$$L = \lim_{n\to\infty} |n+1| \cdot |x| = \lim_{n\to\infty} |n+1| \cdot 0 = \lim_{n\to\infty} 0 = 0.$$

Since $L < 1$ only when $x = 0$, the interval of convergence is just a single point $\{0\}$. (In this case, we say that the Radius of Convergence is $R = 0$.)

8.5 Power Series

EXAMPLE 3 **A power series that converges for all values of x** Find the domain of the Bessel function of order 0 defined by

this means find the interval of convergence.

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Use Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{x^{2n}}\cdot\frac{2^{2n}}{2^{2(n+1)}}\cdot\frac{(n!)(n!)}{(n+1)!\,(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|x^2\cdot\frac{1}{2^2}\cdot\frac{1}{(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{4(n+1)^2}\right|$$

$$= \frac{|x^2|}{4}\lim_{n\to\infty}\frac{1}{(n+1)^2}$$

$$= \frac{x^2}{4}\cdot 0 = 0.$$

Since $L=0<1$ for all x, by the Ratio Test the series $J_0(x)$ converges absolutely for all x. So the interval of convergence is $(-\infty,\infty)$ and the radius of convergence is $R=\infty$.

There is no boundary point to check.