Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Polynomials

Definition. A polynomial in variable x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, \cdots, a_n$ are constants and n is a non-negative integer.

Examples:

$$P(x) = x^3 - 2, \qquad Q(x) = 4(x+1)^3$$

NOT POLYNOMIALS:

$$f(x) = \sqrt{x}, \qquad g(x) = (x-3)^{2/3}$$

Polynomials

Polynomials are one of the simplest functions that we use. They are pleasant to work with in numerical computations because their values may be found by performing a **finite number** of multiplications and additions.

We will show that many functions, such as the exponential and trigonometric functions, can be approximated by polynomials. If the difference between a function and its polynomial approximation is sufficiently small, then we can, for practical purposes, compute with the polynomial in place of the original function.

8.7 Polynomial Approximations to Functions

Let's say an asteroid is moving in space with the position function $f(t)$ where t is time. $f(t)$ is unknown but we can observe the asteroid using a telescope.

- What does $f(0)$ represent? position at $t=0$
- What does $f'(0)$ represent? velocity at $t=0$
- What does $f''(0)$ represent? acceleration at $t=0$
- What does $f^{(n)}(0)$ represent?

If we know everything about the asteroid at time $t = 0$, can we figure out where it will be in 100 years?

Can we rebuild a function from its derivatives?

Approximating a Function from its Derivatives

Suppose we have a function $f(x)$ with $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 1$. Find a polynomial P of degree 2 which agrees with f and its first two derivatives at 0.

P has the general form

$P(x) = a_2 x^2 + a_1 x + a_0$.

We need to satisfy three conditions

$P(0) = f(0)$, $P'(0) = f'(0)$, $P''(0) = f''(0)$.

Observe that $P(0) = 0 + 0 + a_0$

so $a_0 = P(0) = f(0) = 1$.

Hence $a_0 = 1$.

Next we have $P'(x) = 2a_2 x + a_1$

so $P'(0) = 0 + a_1$ and

$a_1 = P'(0) = f'(0) = 1$.

Hence $a_1 = 1$.

Lastly, $P''(x) = 2a_2$ so $2a_2 = P''(0) = f''(0) = 1$.

Hence $2a_2 = 1$

$a_2 = \frac{1}{2}$.

Therefore $P(x) = \frac{1}{2}x^2 + x + 1$.

8.7 Taylor Polynomials

We define the k**th-degree Taylor polynomial of** $f(x)$ **centered at 0** as

n^{th} derivative of $f(x)$ with 0 plugged in

$$T_k(x) = \sum_{n=0}^{k} \frac{\boxed{f^{(n)}(0)}}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k.$$

https://www.desmos.com/calculator/dodhfkxztu

Find the 4th degree Taylor polynomial for $f(x) = e^x$ centered at 0.

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$$

$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{f^{(0)}(0)}{0!} x^0 + \frac{f^{(1)}(0)}{1!} x^1 + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4$$

Now

$f^{(0)}(x) = f(x) = e^x$

$f^{(1)}(x) = f'(x) = e^x$

$f^{(2)}(x) = f''(x) = e^x$

$f^{(3)}(x) = f'''(x) = e^x$

$f^{(4)}(x) = f''''(x) = e^x$

So

$f^{(0)}(0) = e^0 = 1$

$f^{(1)}(0) = e^0 = 1$

$f^{(2)}(0) = e^0 = 1$

$f^{(3)}(0) = e^0 = 1$

$f^{(4)}(0) = e^0 = 1.$

Therefore

$$T_4(x) = \frac{1}{0!} x^0 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4$$

$$\boxed{T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}}$$

Find the 3rd degree Taylor polynomial for $f(x) = \cos x$ centered at 0. $a=0$

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$$

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n = \frac{f^{(0)}(0)}{0!} x^0 + \frac{f^{(1)}(0)}{1!} x^1 + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3$$

$f = \cos x \qquad f(0) = \cos 0 = 1$

$f' = -\sin x \qquad f'(0) = -\sin 0 = 0$

$f'' = -\cos x \qquad f''(0) = -\cos 0 = -1$

$f''' = \sin x \qquad f'''(0) = \sin 0 = 0$

$$T_3(x) = 1 + \frac{0}{1} x + \frac{-1}{2} x^2 + \frac{0}{6} x^3$$
$$= 1 - \frac{1}{2} x^2$$

Remark: Even though $1 - \frac{1}{2}x^2$ is a degree 2 polynomial, it is also the third degree Taylor polynomial for $\cos x$ centered at 0 because the Taylor coefficient for x^3 is 0.

8.7 Taylor Polynomials with Different Centers

We define the k**th-degree Taylor polynomial of** $f(x)$ **centered at** a as

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Find the 3rd degree Taylor polynomial for $f(x) = \cos x$ centered at $\frac{\pi}{4}$. $a = \frac{\pi}{4}$

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(\frac{\pi}{4})}{n!}\left(x-\frac{\pi}{4}\right)^n$$

$$= \frac{f^{(0)}(\frac{\pi}{4})}{0!}\left(x-\frac{\pi}{4}\right)^0 + \frac{f^{(1)}(\frac{\pi}{4})}{1!}\left(x-\frac{\pi}{4}\right)^1 + \frac{f^{(2)}(\frac{\pi}{4})}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{f^{(3)}(\frac{\pi}{4})}{3!}\left(x-\frac{\pi}{4}\right)^3$$

$f^{(0)}(x) = \cos x$

$f^{(1)}(x) = -\sin x$

$f^{(2)}(x) = -\cos x$

$f^{(3)}(x) = \sin x$

$f^{(0)}(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$f^{(1)}(\frac{\pi}{4}) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$f^{(2)}(\frac{\pi}{4}) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

$f^{(3)}(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$= \frac{\frac{\sqrt{2}}{2}}{0!}\left(x-\frac{\pi}{4}\right)^0 + \frac{\frac{-\sqrt{2}}{2}}{1!}\left(x-\frac{\pi}{4}\right)^1 + \frac{\frac{-\sqrt{2}}{2}}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{\frac{\sqrt{2}}{2}}{3!}\left(x-\frac{\pi}{4}\right)^3$$

$$\boxed{T_3(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3}$$