Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

8.1 Sequences and Continuity

Continuity. If the function f is continuous at L and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

In other words, limits can freely move in and out of continuous functions.

8.1 Sequences and Continuity

Find $\lim_{n \to \infty} \sin(\pi/n)$.

First, let's switch to a continuous variable x: $\lim_{x \to \infty} \sin\left(\frac{\pi}{x}\right)$

Observe that $\lim_{x \to \infty} \frac{\pi}{x} = 0$ and $\sin x$ is continuous at $x=0$.

Therefore, by the definition of continuity,

$$\lim_{x \to \infty} \sin(\pi/x) = \sin\left(\lim_{x \to \infty} \pi/x\right)$$

$$= \sin(0)$$

$$= 0.$$

Hence $\lim_{n \to \infty} \sin(\pi/n) = 0$.

8.1 Factorials

The factorial of a positive integer n, denoted by $n!$, is the product of all **positive integers** less than or equal to n. If n is 0, then we define $0! = 1$. $n!$ is undefined if n is negative.

In other words,

$$n! = \begin{cases} n(n-1)\cdots 2\cdot 1 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \\ \text{undefined} & \text{if } n < 0. \end{cases}$$

8.1 Sequences

Show that the sequence $a_n = \dfrac{n!}{n^n}$ converges.

Note that
$$\frac{n!}{n^n} = \frac{\overbrace{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}^{n\text{-terms}}}{\underbrace{n\ \ n\ \ n\ \cdots n\cdot n\cdot n}_{n\text{-terms}}}$$

$$\frac{n!}{n^n} = \frac{1}{n}\left(\underbrace{\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{(n-2)}{n}\cdot\frac{n-1}{n}}\cdot\underbrace{\frac{n}{n}}_{=1}\right)$$

These fractions are less than 1

Hence
$$\frac{n!}{n^n} \le \frac{1}{n}\cdot 1$$

This means
$$\lim_{n\to\infty}\frac{n!}{n^n} \le \lim_{n\to\infty}\frac{1}{n} = 0$$

Since $\frac{n!}{n^n}$ is a non-negative sequence, this means

$$0 \le \lim_{n\to\infty}\frac{n!}{n^n} \quad\text{and}\quad \lim_{n\to\infty}\frac{n!}{n^n} \le 0$$

So $\displaystyle\lim_{n\to\infty}\frac{n!}{n^n} = 0$ by the squeeze thm.

8.1 Sequences

Show that the sequence $a_n = \dfrac{n!}{e^n}$ diverges.

Goal here is to show divergence, so we can compare the original sequence to a simpler one in order to make our conclusion.

Observe that for $n \geq 1$, $\dfrac{1}{3^n} < \dfrac{1}{e^n}$.

Hence for $n \geq 1$, $\dfrac{n!}{3^n} < \dfrac{n!}{e^n}$.

Now let's estimate the numerator.

Observe that $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

For n sufficiently large,

$$\underbrace{n(n-1)(n-2)\cdots 5\cdot 4}_{n-3 \text{ terms}}\cdot 3\cdot 2\cdot 1 > \underbrace{4\cdot 4\cdot 4\cdots 4\cdot 4}_{n-3 \text{ terms}}\cdot 3\cdot 2\cdot 1$$

Here we essentially replaced the numbers greater than 4 with the number 4.

Hence for large n,

$$\frac{4\cdot 4\cdots 4\cdot 3\cdot 2\cdot 1}{3^n} = \frac{4^{n-3}\cdot 3\cdot 2\cdot 1}{3^n} < \frac{n!}{3^n}.$$

Let's take the limit on the left hand side.

$$\lim_{n\to\infty} \frac{4^{n-3}\cdot 3\cdot 2\cdot 1}{3^n} = \lim_{n\to\infty} \frac{4^{n-3}}{3^{n-3}}\cdot\frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3}$$

$$= \frac{6}{27}\lim_{n\to\infty}\left(\frac{4}{3}\right)^{n-3} = \frac{6}{27}\cdot\infty = \infty.$$

Hence

$$\infty = \lim_{n\to\infty}\frac{4^{n-3}\cdot 3\cdot 2\cdot 1}{3^n} \leq \lim_{n\to\infty}\frac{n!}{3^n} \leq \lim_{n\to\infty}\frac{n!}{e^n}$$

So we must have that

$$\lim_{n\to\infty}\frac{n!}{e^n} = \infty.$$

8.1 Sequences

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

In calculus 1, a function f is increasing if $f' > 0$ and decreasing if $f' < 0$.

8.1 Sequences

Show that the sequence $\left\{\frac{3}{n+5}\right\}_{n=1}^{\infty}$ is decreasing.

let's switch to a continuous variable x, which has domain $[1,\infty)$.

$$\frac{d}{dx}\left(\frac{3}{x+5}\right) = \frac{-3}{(x+5)^2} < 0$$

Observe that the derivative above is negative for the x-values in the domain $[1,\infty)$

Hence $\frac{3}{x+5}$ is decreasing for $x \in [1,\infty)$ and we can conclude that $\frac{3}{n+5}$ is decreasing for $n \in \mathbb{N}$.

8.1 Sequences

Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

Switch to the continuous variable x with domain $[1, \infty)$.

Taking the derivative of

$f(x) = \dfrac{x}{x^2+1}$, we get

$$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Observe that $f'(x) < 0$ for $x > 1$.

Hence $f(x)$ is decreasing for $x > 1$

So $a_n = \dfrac{n}{n^2+1}$ is decreasing for $n > 1$.

8.1 Sequences

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

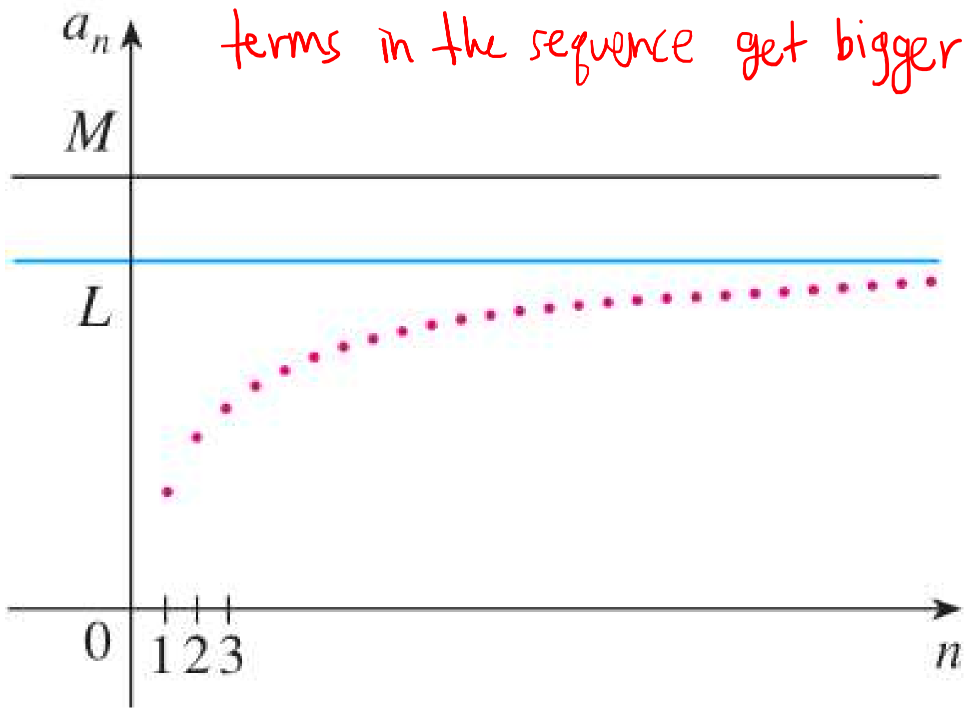
It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

8.1 Sequences

8 **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.



terms in the sequence get bigger but they are bounded by a ceiling so the sequence has to converge.

8.1 Sequences

Find the limit of the sequence $\{a_n\}$ defined by the recurrence relation

$$a_1 = 2 \quad a_{n+1} = \frac{1}{2}(a_n + 6).$$

First, let's verify that this sequence converges.
From the formula, we see that

$a_1 = 2$, $a_2 = 4$, $a_3 = 5$, $a_4 = 5.5$, ...

so the sequence $\{a_n\}$ is positive.
The formula also tells us that the next term in the sequence is the average value between the current term a_n and 6.

Since our initial value $a_1 = 2$ is less than 6, we can safely say that the average values won't exceed 6. Hence

$0 \leq a_n \leq 6$ and the sequence is bounded.

Observe that from our computation, it appears that the sequence is monotonically increasing.

$a_1 < a_2 < a_3 < \ldots$

Therefore by the monotonic sequence theorem, $\{a_n\}$ converges.

8.1 Sequences

Find the limit of the sequence $\{a_n\}$ defined by the recurrence relation

$$a_1 = 2 \quad a_{n+1} = \frac{1}{2}(a_n + 6).$$

Since we verified the convergence of $\{a_n\}$, let's suppose $a_n \to L$ as $n \to \infty$.

Then if we take the limit of the recurrence formula,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6)$$

$$L = \frac{1}{2}\left[\lim_{n\to\infty} a_n\right] + \frac{1}{2}\cdot 6$$

$$L = \frac{1}{2} L + 3$$

Solving for L,

$$\frac{L}{2} = 3$$

$$L = 6.$$

Hence the limit of $\{a_n\}$ is 6.