Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

8.4 “Correct to _ decimal places”

- Let’s say we want to write 1.74 correct to <u>one</u> decimal place.
- Is the rounded answer 1.7?
- How about 1.8?
- Which one is the better answer? Why?
- Estimating a number correct to one decimal place means I want to round the first decimal place.
- This is guaranteed if the distance between the actual number and the rounded number is less than 0.05. (Note that there is <u>one</u> 0 followed by a 5)

8.4 “Correct to x decimal places”

- If we want the rounded answer to be accurate within TWO decimal places, what number should we use to bound the difference between the estimate and the actual number?
- 0.005

- How about THREE decimal places?
- 0.0005

- FOUR decimal places?
- 0.00005

8.4 Alternating Series Estimation Theorem

Alternating Series Estimation Theorem.

If $\sum_{n=1}^{\infty}(-1)^{n-1}\, b_n = s$ is the sum of an alternating series that satisfies

$$\text{(i)} \lim_{k\to\infty} b_k = 0 \qquad \text{and} \qquad \text{(ii)}\ b_k \geq b_{k+1}$$

then $|R_n|$, the error for the n-th partial sum, is less than or equal to the $(n+1)$-th term, b_{n+1}.

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Note that $s_n = \sum_{k=1}^{n}(-1)^{k-1}b_k$. In other words, the error will be less than or equal to the next term.

8.4 Alternating Series Estimation Theorem

V EXAMPLE 4 Using the Alternating Series Estimation Theorem

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

First let's verify that the series converges

Since we see a factorial, let's use the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1}$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0$$

Since $L = 0 < 1$, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges absolutely by the Ratio Test.

Now we know that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges to a number.

Let's estimate the series (an infinite sum) correct to three decimal places.

Since $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ is an alternating series, we use the alternating series estimation theorem, which tells us how good our partial sum is compared to the infinite sum.

Three decimal places mean our error bound is 0.0005.

$$|R_n| \le b_{n+1} < 0.0005$$

$$\frac{1}{(n+1)!} < 0.0005$$

$$\frac{1}{0.0005} < (n+1)!$$

$$\frac{10000}{5} = 2000 < (n+1)!$$

To solve for n, try a few numbers.

$n=5$

$6! = 720$

$n=6$

$7! = 5040$

Since $n=6$ is the first integer that satisfies $(n+1)! > 2000$, we use $n=6$.

Computing the partial sum S_n for $n=6$,

$$S_6 = \sum_{n=0}^{6} \frac{(-1)^n}{n!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$$
$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$
$$= \frac{53}{144} = 0.368056.$$

So in conclusion, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ is approximately 0.368

We will see later that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges to e^{-1}, which is approximately $e^{-1} = 0.367879$. Compare this to our answer 0.368. Our answer is indeed what one would get if he or she rounded 0.367879 to three decimal places. This means we can confidently compute infinite sums to any number of decimal places without knowing what the infinite sum converges to by using approximation theorems.

8.3 Remainder Estimate for the Integral Test.

Suppose $f(k) = a_k$, where $f(x)$ is a continuous, positive decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = s - s_n$ where s_n is the n-th partial sum, then

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx.$$

Also,

$$s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\, dx.$$

8.3 Remainder Estimate for the Integral Test

V EXAMPLE 6 Estimating the sum of a series

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

(a) $$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

The first 10 terms are

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{10^3} \approx 1.1975$$

Since the index starts at 1,

$$1+\frac{1}{2^3}+\cdots+\frac{1}{10^3} = S_{10}.$$

Since the remainder represents the errors involved, Remainder Estimate for the Integral Test gives us

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx.$$

Here n represents the upper index of the partial sum $\sum_{k=1}^{10} \frac{1}{k^3}$.

8.3 Remainder Estimate for the Integral Test

V EXAMPLE 6 Estimating the sum of a series

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Observe that $f(x) = \frac{1}{x^3}$.

Since $n = 10$,

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx.$$

Computing the integrals,

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_{11}^{t} \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{-2t^2} + \frac{1}{2(11)^2}\right] = \frac{1}{242}.$$

Similarly, $\displaystyle\int_{10}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = \frac{1}{200}$.

Hence the error associated with the partial sum S_{10} is bounded in the following way:

$$\frac{1}{242} \le R_{10} \le \frac{1}{200}.$$

8.3 Remainder Estimate for the Integral Test

V EXAMPLE 6 Estimating the sum of a series

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

(b) Here 0.0005 represents the upper bound for our error. Note that we can't solve for the error but we can find equations for the bounds of the error.

This means in general, the accuracy requirement should be on the far-right side of the remainder inequality:

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx \le 0.0005$$

8.3 Remainder Estimate for the Integral Test

V EXAMPLE 6 Estimating the sum of a series

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Since we know from part (a) that

$$\int_n^\infty f(x)\,dx = \frac{1}{2n^2},$$

we have

$$R_n \leq \frac{1}{2n^2} \leq 0.0005.$$

Solving for the first n that satisfies the inequality

$$\frac{1}{2n^2} \leq 0.0005,$$

$$\frac{1}{2(0.0005)} \leq n^2$$

$$\sqrt{1000} \leq n$$

$$31.62 \leq n.$$

But n must be an integer so n=32 is the first integer that satisfies the above inequality. Therefore we need 32 terms to ensure that the sum is accurate to within 0.0005.