Daily Quiz

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8.4 Absolute Convergence

Given any series $\sum_{n=1}^{\infty} a_n$, we can consider the corresponding series of absolute values

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

Definition. A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Example. A series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

Remark. Notice that if $\sum_{n=1}^{\infty} a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence for positive series.

8.4 Example of Absolutely Convergent Series

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$ **converges** as an alternating series by the alternating series test.

But it also **converges absolutely** because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2$.

Absolute Convergence is Stronger than Convergence

Theorem. If a series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**, then it is **convergent**.

Conditional Convergence

Definition. A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is **not absolutely convergent** but still **converges**.

Version for Alternating Series

Theorem. If an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is **absolutely convergent**, then it is **convergent**.

Conditional Convergence for Alternating Series

Definition. An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is called **conditionally convergent** if it is **not absolutely convergent** but still **converges**.

8.4 Conditional Convergence

Examples of conditionally convergent series:

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges but its series of absolute values $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. (Why?)

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$ converges but its series of absolute values $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges. (Why?)

8.4 Modes of Convergence

Absolutely Convergent

Definition. A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Conditionally Convergent

Definition. A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is <u>not</u> absolutely convergent but still converges.

Divergent

Definition. A series $\sum_{n=1}^{\infty} a_n$ is divergent if the sequence of its partial sums $s_m = \sum_{n=1}^{m} a_n$ has no limit as n goes to infinity.

8.4 Absolute Convergence

$-1 \le \cos n \le 1$

$0 \le |\cos n| \le 1$

EXAMPLE 7 Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Test for absolute convergence.

abs $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$

Since $|\cos n| \ge 0$ for all n,

$\dfrac{|\cos n|}{n^2} \ge 0$ for all n.

Since $\dfrac{1}{n^2} \ge 0$ for all n and we have

$\dfrac{|\cos n|}{n^2} \le \dfrac{1}{n^2}$,

we can apply the Direct Comparison Test

so

$$\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges via p-test $(p=2)$, the series $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ converges.

Therefore $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely.

8.4 The Ratio Test

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

8.4 The Ratio Test

Using the Ratio Test Test the series $\sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{n^3}{3^n}}_{a_n}$ for absolute convergence.

$$\frac{1}{3^{n+1}} = \frac{1}{3^n \cdot 3^1}$$

use the ratio test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right|$$

$$= \lim_{n\to\infty} \left[\frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} \right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{n+1}{n} \right)^3 \cdot \frac{1}{3} \right] = \frac{1}{3} \lim_{n\to\infty} \left[\left(1 + \frac{1}{n} \right)^3 \right]$$

$$= \frac{1}{3} \left(1 + \frac{1}{\lim_{n\to\infty} n} \right)^3$$

$$= \frac{1}{3} \left(1 + \frac{1}{\infty} \right)^3$$

$$= \frac{1}{3} = L$$

Since $L = \frac{1}{3} < 1$, by the Ratio Test, $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is absolutely convergent.

8.4 The Ratio Test

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Use the ratio test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n \cdot (n+1)}{n^n} \cdot \frac{1}{n+1}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n$$

$$= \text{“}1^{\infty}\text{”} \text{ indeterminate form}$$

use logarithmic L'Hospital's technique.

$$\ln L = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)$$

$$= \lim_{n\to\infty} \ln\left(\left(1+\frac{1}{n}\right)^n\right) \quad \text{(by continuity of ln)}$$

$$= \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)$$

$$= \text{“}\infty \cdot 0\text{”} \quad \text{(Indeterminate)}$$

use one of the techniques for L'Hospital's Rule

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

$$= \text{“}\frac{0}{0}\text{”} \quad \text{(Indeterminate)}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \cdot -\frac{1}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\frac{1}{\infty}} = 1.$$

Hence $\ln L = 1$ so

$$L = e^1 = e.$$

Since $L = e > 1$, by the Ratio Test

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges.}$$

Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$$

There are a lot of exponents so let's use the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{10^{n+1}}{(n+2)\,4^{2(n+1)+1}} \frac{(n+1)\,4^{2n+1}}{10^n}$$

$$= \lim_{n\to\infty} \frac{10^{n+1}}{10^n} \cdot \frac{n+1}{n+2} \cdot \frac{4^{2n+1}}{4^{2n+3}}$$

$$= \lim_{n\to\infty} 10 \cdot \frac{n+1}{n+2} \cdot \frac{1}{16}$$

$$= \frac{10}{16} \cdot \lim_{n\to\infty} \frac{n+1}{n+2}$$

$$= \frac{10}{16} \cdot 1$$

So $L = \frac{10}{16} < 1$ and by the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ converges absolutely.

Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$$

We must first test for absolute convergence because the definition of conditional convergence requires a series to fail absolute convergence.

We have $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ as the absolute value version of the original series.

Now $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent via the p-test $(p=\frac{1}{2}<1)$ so we immediately see that $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ is not absolutely convergent.

Since our series is an alternating series, let's see if it converges using the alternating series test.

Checking the hypotheses of the AST:

① vanishing at infinity

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

② Decreasing

$$b_n = \frac{1}{\sqrt{n}}, \quad f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

$$f'(x) = -\frac{1}{2} \cdot x^{-\frac{3}{2}} = \frac{-1}{2x^{3/2}}$$

Since $f'(x) < 0$ for $x \geq 1$,

f is decreasing and so is

$b_n = \frac{1}{\sqrt{n}}$ for $n \geq 1$.

In conclusion, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges by the AST.

Since $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges but is not absolutely convergent, it is <u>conditionally convergent</u>.

Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

We see a factorial so let's use the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{100^n}{100^{n+1}}$$

$$= \lim_{n\to\infty} (n+1) \cdot \frac{1}{100}$$

$$= \frac{1}{100} \cdot \lim_{n\to\infty} (n+1)$$

$$= \infty$$

Since $L > 1$, the series $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ <u>diverges by the Ratio Test.</u>