We define the **factorial** for positive integers by

$$0! = 1 \qquad \text{and} \qquad n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n \text{ for } n \geq 1.$$

Computing.
We compute the first few factorials:

$$\begin{aligned}
0! &= 1\\
1! &= 1\\
2! &= 1 \cdot 2 = 2\\
3! &= 1 \cdot 2 \cdot 3 = 6\\
4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24\\
5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120
\end{aligned}$$

Note that in order to get to 4! from 3!, we need only to multiply 3! by four:

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = (1 \cdot 2 \cdot 3) \cdot 4 = 3! \cdot 4.$$

This is a specific case of the general fact that

$$n! = (n-1)! \cdot n.$$

Simplifying.
By writing out the terms of each factorial, we can simplify expressions that involve multiple factorials in the numerator and denominator of a fraction. For example,

$$\frac{6!}{4! \cdot 5!} = \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot 6}{(1 \cdot 2 \cdot 3 \cdot 4) \cdot (\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5})} = \frac{6}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{\cancel{6}}{\cancel{6} \cdot 4} = \frac{1}{4}$$

Another example:

$$\frac{10!}{5!} = \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5}} = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 30,240$$

Here's another example. In this one, the factorials are too long to write out completely, so we use dots ("$\cdots$") to indicate the continuing product. It's useful to write down the first few terms and the last few terms.

$$\begin{aligned}
\frac{100!}{97!} &= \frac{1 \cdot 2 \cdot 3 \cdots 96 \cdot 97 \cdot 98 \cdot 99 \cdot 100}{1 \cdot 2 \cdot 3 \cdots 96 \cdot 97}\\
&= \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdots \cancel{96} \cdot \cancel{97} \cdot 98 \cdot 99 \cdot 100}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdots \cancel{96} \cdot \cancel{97}}\\
&= 98 \cdot 99 \cdot 100\\
&= 970,200.
\end{aligned}$$

Of course, what we really want to do is to simplify expressions with some general positive integer n in them.

Math 2300 Practice with Factorials Name:

First, if we have a factorial with some variable inside it, such as $(2n)!$, and we want to expand this factorial, we will *always* use the dots ("$\cdots$") to indicate a continuing product between some beginning and ending terms. So for example, we could write

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n$$

where the dots indicate that we are also multiplying with all of the numbers between 4 and $n-2$. Another example: we could write

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-2) \cdot (2n-1) \cdot (2n)$$

where the dots indicate that we are also multiplying with all of the numbers between 4 and $2n-2$. It's up to you how many beginning and ending terms you want to write out.

How do we simplify fractions that involve factorials with n in them? The first step is to write out each factorial product, including a few beginning and ending terms. For example,

$$\frac{(n+2)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n+1) \cdot (n+2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n}$$

It's a good idea to try to get the ending terms to match up at some point. To do this, we'll write down a couple more ending terms of the product in the numerator:

$$\frac{(n+2)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n \cdot (n+1) \cdot (n+2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n}$$

Now we cancel just like we did with numbers:

$$\frac{(n+2)!}{n!} = \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdots \cancel{(n-1)} \cdot \cancel{n} \cdot (n+1) \cdot (n+2)}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdots \cancel{(n-1)} \cdot \cancel{n}} = (n+1)(n+2).$$

You can use this technique along with algebra to simplify more complicated expressions:

$$\begin{aligned}
\frac{(n!)^2}{(n-1)!(n+1)!} &= \frac{n! \cdot n!}{(n-1)!(n+1)!} \\
&= \frac{[1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n] \cdot [1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n]}{[1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1)] \cdot [1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n \cdot (n+1)]} \\
&= \frac{[\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdots \cancel{(n-2)} \cdot \cancel{(n-1)} \cdot n] \cdot [1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n]}{[\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdots \cancel{(n-2)} \cdot \cancel{(n-1)}] \cdot [1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n \cdot (n+1)]} \\
&= \frac{n \cdot [1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n]}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n \cdot (n+1)]} \\
&= \frac{n \cdot [\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdots \cancel{(n-2)} \cdot \cancel{(n-1)} \cdot \cancel{n}]}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdots \cancel{(n-2)} \cdot \cancel{(n-1)} \cdot \cancel{n} \cdot (n+1)]} \\
&= \frac{n}{n+1}.
\end{aligned}$$

Exercises.

1. Write each long expression as a single factorial.

   (a) $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$

   (b) $1 \cdot 2 \cdot 3 \cdots 100$

   (c) $1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1) \cdot n$

   (d) $1 \cdot 2 \cdot 3 \cdots 56 \cdot 57 \cdot 58$

   (e) $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (5n-4) \cdot (5n-3) \cdot (5n-2)$

2. Write each factorial as a long product, using "$\cdots$" as necessary. Include at least two beginning and at least two ending terms.

   (a) $8!$

   (b) $45!$

   (c) $(n+1)!$

   (d) $(2n)!$

   (e) $(2n+5)!$

3. Simplify the following expressions.

   (a) $\dfrac{5!}{6!}$

   (b) $\dfrac{7!}{5!}$

   (c) $\dfrac{(3!)^2}{9}$

   (d) $\dfrac{3! \cdot 4!}{5!}$

   (e) $\dfrac{9!}{5! \cdot 3!}$

   (f) $\dfrac{(4-1)!}{4!}$

   (g) $\dfrac{(2 \cdot 3)!}{3!}$

   (h) $\dfrac{88!}{90!}$

   (i) $\dfrac{77! \cdot 2!}{78!}$

   (j) $\dfrac{38! \cdot 3!}{39!}$

4. Simplify the following expressions.

   (a) $\dfrac{(n+2)!}{n!}$

   (b) $\dfrac{n}{n!}$

   (c) $\dfrac{(n-1)! \cdot n!}{(n!)^2}$

   (d) $\dfrac{(n+5)!}{(n+1)!}$

   (e) $\dfrac{((n+1)!)^3}{(n!)^3}$

   (f) $\dfrac{(n^2-1)!}{(n^2)!}$

   (g) $\dfrac{(2n)!}{(2n-2)! \cdot 2!}$