Collaborative Quiz 2. 40 points total. **Due 2:00 PM Friday, October 19th, 2018.**

Your Name: Solutions

Teammates' Names: ____

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Instructions: **Must be 4 people per group; no more, no less.** Provide **detailed solutions** to the attached problems and have one person with the best handwriting submit their work. **You must cite all the theorems you use and check the hypotheses.** As part of the Honor Code, make sure to teach your fellow teammates how to solve each problem.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total

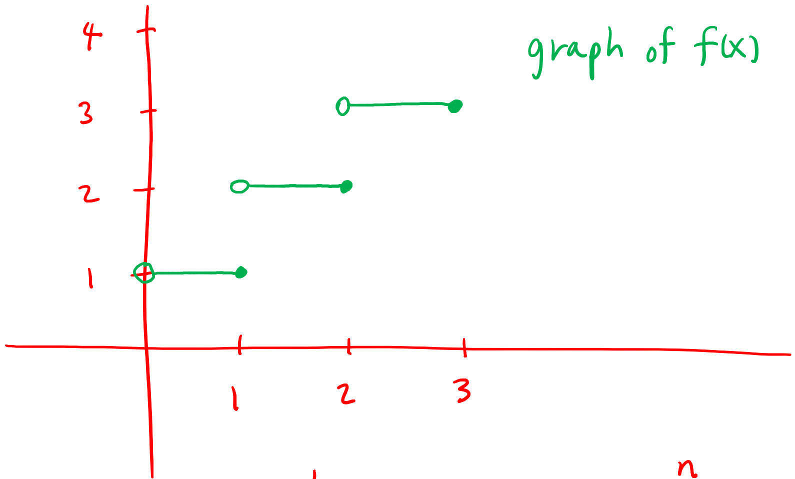
By the Honor Code, I will not plagiarize and I will help the others in the group understand the questions.

SIGN: ____

DATE: ____________________________

1. (3 points) Let f be the ceiling function $f(x) = \lceil x \rceil$, which is defined as the least integer greater than or equal to x. For example, $\lceil 2.4 \rceil = 3$.

Compute the average value of the ceiling function f on the interval $[0, n]$ where n is a positive integer.



$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{n-0}\int_0^n \lceil x \rceil\,dx$$

observe that the interval $[0,n]$ can be broken up into n pieces.

$$\frac{1}{n}\int_0^n \lceil x \rceil\,dx = \frac{1}{n}\left[\int_0^1 \lceil x \rceil dx + \int_1^2 \lceil x \rceil dx + \cdots + \int_{n-1}^n \lceil x \rceil dx\right]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1} \lceil x \rceil\,dx$$

Since $f(x) = \lceil x \rceil$ is constant on each interval $[k, k+1)$,

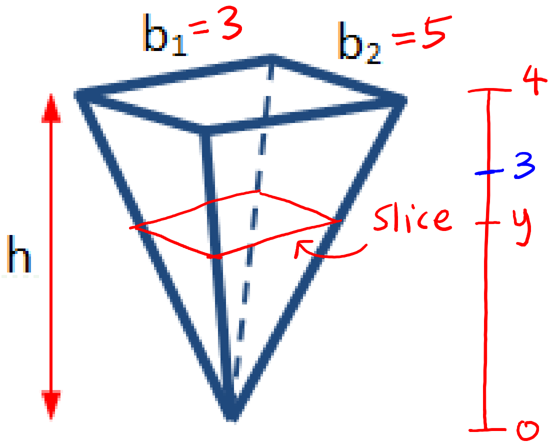
$$f_{avg} = \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1}\lceil x \rceil dx = \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1}(k+1)\,dx = \frac{1}{n}\sum_{k=0}^{n-1}(k+1)\int_k^{k+1} dx$$

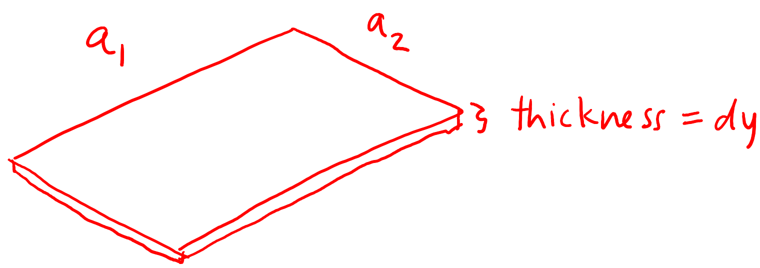
$$= \frac{1}{n}\sum_{k=0}^{n-1}(k+1)\big[x\big]_k^{k+1} = \frac{1}{n}\sum_{k=0}^{n-1}(k+1)\big[(k+1)-k\big]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}(k+1)[1] = \frac{1}{n}\sum_{k=0}^{n-1} k+1 = \frac{1}{n}\big[1+2+3+\cdots+(n-1)+n\big]$$

$$f_{avg} = \frac{1}{n}\,\frac{n(n+1)}{2} = \frac{n+1}{2}$$

2. (4 points) Suppose a tank in the shape of an inverted rectangular pyramid with $h = 4$ meter, $b_1 = 3$ meter, and $b_2 = 5$ meter is filled with water to a height of 3 meter.
Set up, but do not evaluate, the integral representing the work required to empty the tank by pumping all of the water to the top of the tank.
Use ρ for the density of water and g for the gravitational acceleration.





Volume of slice $= a_1 a_2\, dy$

Weight of slice $= \rho \cdot g \cdot$ Volume

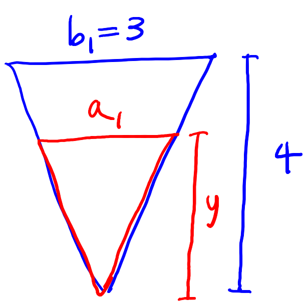
$= \rho g a_1 a_2\, dy$

work done in moving the slice to the top of the tank = weight · distance to the top

$= (\rho g a_1 a_2\, dy)\cdot(4-y)$

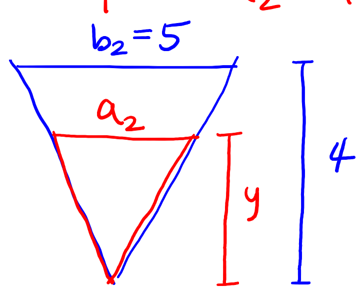
$$\text{Total Work} = \int_0^3 \rho g a_1 a_2 (4-y)\, dy$$

Use similar triangles to find a_1 and a_2 in terms of y.



$\frac{3}{4} = \frac{a_1}{y}$

$a_1 = \frac{3}{4}y$



$\frac{5}{4} = \frac{a_2}{y}$

$a_2 = \frac{5}{4}y$

$$\text{Total Work} = \int_0^3 \rho g \left(\frac{3}{4}y\right)\left(\frac{5}{4}y\right)(4-y)\, dy \quad \text{Joules}$$

3. (5 points) Consider the series below:

$$2 - \frac{2}{5} + \frac{2}{25} - \frac{2}{125} + \frac{2}{625} - \cdots$$

(a) Rewrite the series using the sigma notation.

$n=0$ $n=1$ $n=2$ $n=3$

$\frac{2}{1}$ $-\frac{2}{5}$ $\frac{2}{25}$ $-\frac{2}{125}$

$$a_n = (-1)^n \frac{2}{5^n}$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n \frac{2}{5^n}$$

(b) Determine whether the series converges or diverges. **If the series converges, find its sum.** Verify the hypotheses of the tests used.

$$\sum_{n=0}^{\infty} (-1)^n \frac{2}{5^n} = \sum_{n=0}^{\infty} 2\left(\frac{-1}{5}\right)^n$$

The series above is a geometric series.

Initial value $a = 2$

Common ratio $r = \frac{a_{n+1}}{a_n} = -\frac{1}{5}$

Since $|r| = \left|-\frac{1}{5}\right| = \frac{1}{5} < 1$, the geometric series <u>converges</u> and

$$\sum_{n=0}^{\infty} 2\left(\frac{-1}{5}\right)^n = \frac{a}{1-r} = \frac{2}{1-\left(-\frac{1}{5}\right)} = \frac{2}{\frac{6}{5}} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

4. (3 points) Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{n}\right)$$

This series doesn't look like a p-series or a geometric series. Let's try to determine the long-term behavior of this series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos\left(\frac{\pi}{\infty}\right) = \cos(0) = 1$$

The above limit shows that in the long run, the series adds 1 infinitely many times.

Hence by the Divergence Test, since $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{n}\right)$ diverges.

5. (5 points) Consider the series $\sum_{n=1}^{\infty} \ln\left(\frac{2n-1}{2n+1}\right)$.

(a) Write out the partial sum s_3. You do not have to simplify.

$$S_3 = \sum_{n=1}^{3} \ln\left(\frac{2n-1}{2n+1}\right) = \ln\left(\frac{1}{3}\right) + \ln\left(\frac{3}{5}\right) + \ln\left(\frac{5}{7}\right)$$

(b) Determine whether the series converges or diverges.

Telescoping sum.

Strategy: Compute the general partial sum and take the limit.

$$S_m = \sum_{n=1}^{m} \ln\left(\frac{2n-1}{2n+1}\right) = \sum_{n=1}^{m} \left(\ln(2n-1) - \ln(2n+1)\right)$$

$$= \left(\ln 1 - \cancel{\ln 3}\right) + \left(\cancel{\ln 3} - \cancel{\ln 5}\right) + \left(\cancel{\ln 5} - \cancel{\ln 7}\right)$$

$$+ \cdots + \left(\cancel{\ln(2m-1)} - \ln(2m+1)\right)$$

$$= \ln 1 - \ln(2m+1) = 0 - \ln(2m+1) = -\ln(2m+1)$$

Therefore $\sum_{n=1}^{\infty} \ln\left(\frac{2n-1}{2n+1}\right) = \lim_{m\to\infty} S_m = \lim_{m\to\infty} -\ln(2m+1)$

$$= -\ln(\infty) = -\infty$$

Hence the series $\sum_{n=1}^{\infty} \ln\left(\frac{2n-1}{2n+1}\right)$ diverges.

6. (3 points) Determine whether the series converges or diverges.

$$\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}} + \frac{1}{3\sqrt{4}} + \frac{1}{4\sqrt{5}} + \frac{1}{5\sqrt{6}} + \cdots$$

In sigma notation the series is $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$

Since $\frac{1}{n\sqrt{n+1}} \sim \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ and the series

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-test ($p = 3/2 > 1$),

we use the Limit Comparison Test.

Check the hypothesis:

Positivity: Since both $\frac{1}{n\sqrt{n+1}}$ and $\frac{1}{n\sqrt{n}}$ are positive for $n \geq 1$, the hypothesis is satisfied.

Taking the Limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n+1}}}{\frac{1}{n\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$

(by continuity of $\sqrt{}$) $= \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \sqrt{1} = 1.$

Since the limit exists and is non-zero, by the Limit Comparison Test either both $\sum \frac{1}{n\sqrt{n+1}}$ and $\sum \frac{1}{n\sqrt{n}}$ converge or both diverge. But $\sum \frac{1}{n\sqrt{n}}$ converges by p-test ($p = \frac{3}{2} > 1$) so $\sum \frac{1}{n\sqrt{n+1}}$ also converges.

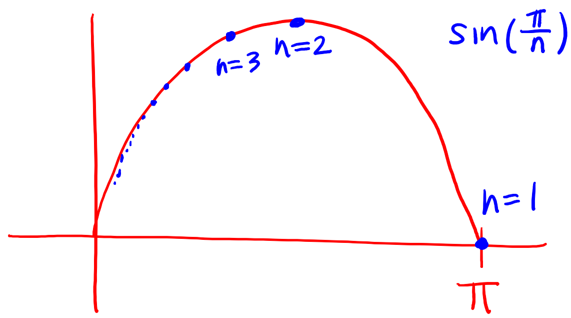
7. (3 points) Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right)$$

Observe that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$. Let $x = \frac{\pi}{n}$. Then $n = \frac{\pi}{x}$ and n approaches $+\infty$ as x approaches 0 from the right. Hence $n \to \infty$ as $x \to 0^+$ and we can use change of variables to get $\lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = 1$. This suggests that the Limit Comparison Test might work. But first we need to verify the hypothesis for the Limit Comparison Test.

Check the hypothesis:

Positivity: $\frac{\pi}{n}$ is positive for $n \geq 1$ but it's not clear for $\sin\left(\frac{\pi}{n}\right)$. The graph of $\sin x$ suggests that $\sin x \geq 0$ for $x = \pi, \frac{\pi}{2}, \frac{\pi}{3}, \ldots$



Hence $\sin\left(\frac{\pi}{n}\right)$ is also positive for $n \geq 1$ and the hypothesis is satisfied.

Taking the Limit: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = 1$.

Since the limit exists and is non-zero, by the Limit Comparison Test either both $\sum \sin\left(\frac{\pi}{n}\right)$ and $\sum \frac{\pi}{n}$ converge or both diverge. But $\sum_{n=1}^{\infty} \frac{\pi}{n} = \pi \sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test ($p = 1$) so $\sum_{n=1}^{\infty} \frac{\pi}{n}$ diverges. Hence $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right)$ also diverges.

8. (5 points)

(a) Show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

③ Positivity
$f(x) = \frac{1}{x \ln x} > 0$ for $x \geq 2$

Integral Test. Let $f(x) = \frac{1}{x \ln x}$.

Check the hypotheses:

① Continuity: Since $f'(x) = \frac{-(1+\ln x)}{(x \ln x)^2}$, f is differentiable and continuous for $x > 1$.

② Decreasing: Since $f'(x) = \frac{-(1+\ln x)}{(x \ln x)^2} < 0$ for x in $[2, \infty)$, f is decreasing for $x \geq 2$.

Computing the integral: $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{u(2)}^{u(\infty)} \frac{du}{u}$ $\quad u = \ln x$, $du = \frac{dx}{x}$

$$= \lim_{t \to \infty} \int_{\ln 2}^{t} \frac{du}{u} = \lim_{t \to \infty} \Big[\ln|u|\Big]_{\ln 2}^{t} = \lim_{t \to \infty} \Big[\ln|t| - \ln|\ln 2|\Big] = \infty.$$

Hence $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx$ diverges.

By the integral test, since $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx$ diverges, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ also diverges.

(b) Show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n!)}$ diverges.

Direct Comparison Test. Compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

Check the hypothesis:

Positivity: Both $\frac{1}{\ln(n!)}$ and $\frac{1}{n \ln n}$ are positive for $n \geq 2$.

Comparing the terms: Observe that $n(n-1) \cdots 2 \cdot 1 = n! \leq n^n = n \cdot n \cdot \cdots \cdot n$ for $n \geq 2$. Then since $\ln x$ is an increasing function, $\ln(n!) \leq \ln(n^n) = n \ln(n)$. But then $\frac{1}{\ln(n!)} \geq \frac{1}{n \ln(n)}$ so we have $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \leq \sum_{n=2}^{\infty} \frac{1}{\ln(n!)}$. By the Direct Comparison Test, since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n!)}$ also diverges.

9. (6 points) Consider the alternating series below:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt[4]{n}}$$

(a) Determine whether the series is conditionally convergent, absolutely convergent, or divergent.

First test for absolute convergence. The series of absolute values is $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$. Since this is a p-series with $p=\frac{1}{4}$, by the p-test $\sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$ is divergent. Hence $\sum (-1)^{n-1}\frac{1}{n^{1/4}}$ is not absolutely convergent.

Let's test for convergence using the Alternating Series Test.

Check the hypotheses:

① vanishing at infinity

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^{1/4}} = 0$$

② Decreasing

$$b_n = \frac{1}{n^{1/4}}, \quad f(x) = \frac{1}{x^{1/4}}, \quad f'(x) = -\frac{1}{4}x^{-5/4} < 0$$

for $x \geq 1$.

By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^{1/4}}$ converges.

Since the series converges but is not absolutely convergent, it is conditionally convergent.

(b) Use the Alternating Series Estimation Theorem to determine how large n must be in order to guarantee that the estimation error $|R_n|$ is less than or equal to $\frac{1}{10}$.

$$|R_n| = |s - s_n| \leq b_{n+1}$$

$$|R_n| \leq b_{n+1} \leq \frac{1}{10}$$

$$|R_n| \leq \frac{1}{(n+1)^{1/4}} \leq \frac{1}{10}$$

$$10 \leq (n+1)^{1/4}$$

$$10^4 \leq (n+1)$$

$$10000 \leq n+1$$

$$9999 \leq n$$

n must be at least 9999.

10. (3 points) Determine whether the series is conditionally convergent, absolutely convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-2)^n n!}{(2n)!}$$

Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1} (n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{(-2)^n \, n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(-2)^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \left| (-2) \cdot (n+1) \cdot \frac{1}{(2n+2)(2n+1)} \right|$$

$$= \lim_{n\to\infty} \left| (-1) \frac{2n+2}{(2n+2)(2n+1)} \right| = \lim_{n\to\infty} \frac{1}{2n+1} = 0.$$

Since $L = 0 < 1$, $\sum_{n=1}^{\infty} \frac{(-2)^n \, n!}{(2n)!}$ converges absolutely by the Ratio Test.