Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Taylor's Inequality

Suppose $T_k(x)$ is a Taylor polynomial centered at a for the function f. Let d be a constant and $|f^{(k+1)}(x)| \leq M$ for values of x satisfying $|x-a| \leq d$. Then for those values of x, the error $R_k(x)$ of the Taylor polynomial $T_k(x)$ satisfies the inequality

$$|R_k(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1} \leq \frac{M}{(k+1)!}d^{k+1}$$

In other words, the error from $T_k(x)$ is bounded by some constants;

$$|R_k(x)| \leq \frac{M}{(k+1)!}d^{k+1}$$

Deciphering Taylor's Inequality:

1. $|x-a| \leq d$ looks **very similar** to the inequality $|x-a| < R$ (R is the radius of convergence.)

2. a is the **center** of the Taylor polynomial, and it is the center of the intervals.

3. d is the **radius of approximation**, which is the distance from the center to the boundary of the **interval of approximation**. In order for the approximation to make sense, d must be less than R:

$$d < R.$$

4. M is computed by **maximizing** $|f^{(k+1)}(x)|$ in the interval of approximation $[a-d, a+d]$. (Usually maximizing an increasing, decreasing, or an oscillating function. Techniques like the Closed Interval Method can be used.)

Controlling the Error

There are three moving parts to Taylor's Inequality:

1. k, the degree of the Taylor polynomial
2. d, the radius of approximation.
3. M, the maximum bound for the $(k+1)$-th derivative of $f(x)$ inside the interval of approximation.

The last moving part M **is dependent on both** k **and** d since the maximum of the $(k+1)$-th derivative is taken over the interval $[a-d, a+d]$.

The **error gets smaller** $(|R_k| \to 0)$ as one either

1. Increases the degree k of the Taylor polynomial $(k \to \infty)$ or
2. Reduces the size of the interval of approximation $(d \to 0)$.

Desmos Examples to Play With

Taylor Polynomials of degree k and the radius of approximation d:

https://www.desmos.com/calculator/ljbm9jewu0

Graphs of the Taylor polynomials and the errors for various functions:

https://www.cengage.com/math/discipline_content/stewartccc4/2008/14_cengage_tec/publish/deployments/concepts_4e/4c3_tool.html#

Four ways to use Taylor's Inequality - #1

1. Showing that a Taylor series converges to its function f.

Prove that $f(x) = e^x$ is equal to its Maclaurin series $T(x)$.

Prove that $f(x) = \sin x$ is equal to its Maclaurin series $T(x)$.

Four ways to use Taylor's Inequality - #2

2. Determining accuracy; find a bound for the error.

1. Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

2. How accurate is this approximation when $7 \leq x \leq 9$?

Four ways to use Taylor's Inequality - #3

3. The interval of approximation is unknown; solve for the radius of approximation d and use it to find the interval of approximation.

1. What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$? Use this to approximate $\sin 12°$.

2. For what values of x is this approximation accurate to within 0.00005?

Four ways to use Taylor's Inequality - #4

4. The degree k of the Taylor polynomial needs to be at least some number.

Let $T_k(x)$ be the Taylor polynomial centered at 0 for $f(x) = e^x$. Use Taylor's Inequality to determine the degree k that should be used to estimate the number e^1 with an error less than 0.6.