Daily Quiz

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Approximation Methods for Definite Integrals

When approximating a definite integral $\int_a^b f(x)\,dx$, we rely on integration using power series and apply one of the two methods below:

1. **Integral Test Remainder Estimate**
2. **Alternating Series Remainder Estimate**

(a) Evaluate $\int \frac{1}{1+x^7}dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5} \frac{1}{1+x^7}dx$ correct to within 10^{-7}.

(a) $\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty}(-x^7)^n = \sum_{n=0}^{\infty}(-1)^n x^{7n}$; $\int \frac{1}{1+x^7}dx = \int \sum_{n=0}^{\infty}(-1)^n x^{7n}dx = \sum_{n=0}^{\infty}\int (-1)^n x^{7n}dx$

$= \sum_{n=0}^{\infty}(-1)^n \int x^{7n}dx = \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{7n+1}}{7n+1}\right) + C$

(b) $\int_0^{0.5}\frac{1}{1+x^7}dx = \sum_{n=0}^{\infty}(-1)^n\left[\frac{x^{7n+1}}{7n+1}\right]_0^{0.5} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{(0.5)^{7n+1}}{7n+1} - \frac{0}{7n+1}\right]$

$= \sum_{n=0}^{\infty}(-1)^n\frac{(0.5)^{7n+1}}{7n+1}$

Our goal is to approximate the above series using techniques that we learned, including the Alternating Series Estimation Theorem and the Remainder Estimate for the Integral Test.

Since we have an alternating series, let's use the Alternating Series Estimation Theorem.

To use the ASET, we need to first verify that $\sum_{n=0}^{\infty} (-1)^n \frac{(0.5)^{7n+1}}{7n+1}$ converges.

using the Ratio Test,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(0.5)^{7(n+1)+1}}{7(n+1)+1} \cdot \frac{7n+1}{(-1)^n (0.5)^{7n+1}} \right| = \lim_{n\to\infty} \frac{7n+1}{7n+8} \frac{(0.5)^{7n+8}}{(0.5)^{7n+1}}$$

$= (0.5)^7 = \left(\frac{1}{2}\right)^7 = \frac{1}{128}$. Since $L = \frac{1}{128} < 1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{(0.5)^{7n+1}}{7n+1}$ converges absolutely.

Now the ASET gives

$$|s - s_n| = |R_n| < b_{n+1} \leq 10^{-7}$$

$$|s - s_n| < \frac{(0.5)^{7(n+1)+1}}{7(n+1)+1} \leq 10^{-7}$$

$$\frac{(0.5)^{7n+8}}{7n+8} \le 10^{-7}.$$ To solve the inequality, we try a few values of n.

$n=1$ $$\frac{(0.5)^{7+8}}{7+8} = 2.03 \times 10^{-6}$$

$n=2$ $$\frac{(0.5)^{14+8}}{14+8} = 1.08 \times 10^{-8}$$

This shows that $n=2$ is the first integer that satisfies $b_{n+1} \le 10^{-7}$.

Therefore n needs to be greater than or equal to 2 in order for the partial sum $S_n = \sum_{k=0}^{n} (-1)^k \frac{(0.5)^{7k+1}}{7k+1}$ to be correct within 10^{-7}.

Since we need to compute the approximation,

$$\int_0^{0.5} \frac{1}{1+x^7}dx \approx S_2 = \sum_{n=0}^{2} (-1)^n \frac{(0.5)^{7n+1}}{7n+1} = 0.5 - \frac{(0.5)^8}{8} + \frac{(0.5)^{15}}{15} = 0.499514$$

1. Find $\int e^{-x^2}\,dx$ as a power series.

2. Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

① Use the Taylor series of e^x to obtain e^{-x^2} as a power series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Therefore

$$\int e^{-x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{n!}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}dx$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}\right) + C$$

② Using the result from the previous part,

$$\int_0^1 e^{-x^2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{x^{2n+1}}{2n+1}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(2n+1)}$$

Hence it suffices to estimate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(2n+1)}$ correct to within an error of 0.001. Since the series is an alternating series, we use the Alternating Series Estimation Theorem. But first let's make sure $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(2n+1)}$ converges.

Using the Ratio Test,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(n+1)!\,(2(n+1)+1)} \cdot \frac{n!\,(2n+1)}{(-1)^n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{(2n+1)}{(2n+3)}$$

$$= \lim_{n\to\infty} \frac{2n+1}{(n+1)(2n+3)} = 0.$$

Since $L = 0 < 1$, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(2n+1)}$ converges absolutely.

Now let's apply the ASET.

$$|s - s_n| = |R_n| < b_{n+1}$$

$$|s - s_n| < b_{n+1} = \frac{1}{(n+1)!} \frac{1}{2(n+1)+1} \leq 0.001$$

$$\frac{1}{(n+1)!\,(2n+3)} \leq 0.001$$

Try $n=3$

$$\frac{1}{(3+1)!\,(2\cdot 3+3)} = 0.00463$$

Try $n=4$

$$\frac{1}{(4+1)!\,(2\cdot 4+3)} = 0.000758 \leq 0.001$$

This shows that $n=4$ is the smallest integer in which the inequality

$$b_{n+1} \leq 0.001 \quad \text{holds.}$$

Therefore n needs to be at least 4 in order for the partial sum s_n to be correct within an error of 0.001.

Hence

$$\int_0^1 e^{-x^2}dx \approx s_4 = \sum_{n=0}^{4} \frac{(-1)^n}{n!} \frac{1}{(2n+1)}$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216}$$

$$= 0.747487$$

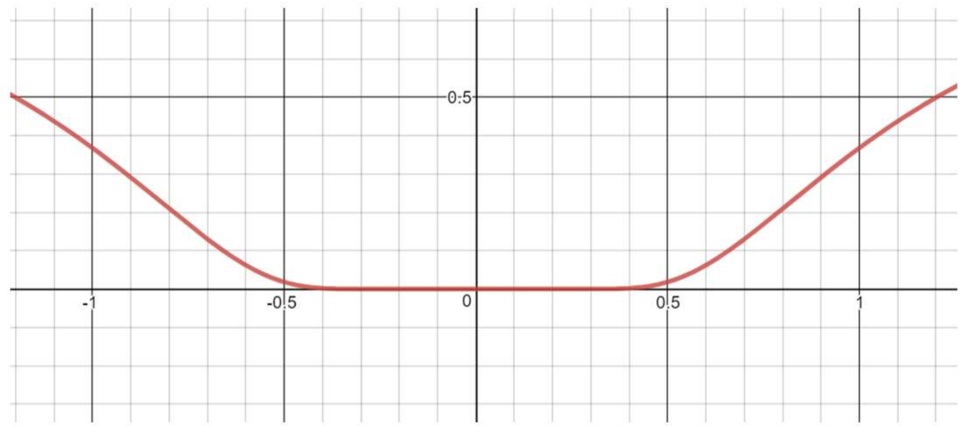
and so

$$\int_0^1 e^{-x^2}dx \approx s_4 = 0.747487$$

Is a function $f(x)$ really equal to its Taylor series **inside its interval of convergence**?

Not always.

Consider the function



$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

$f(x)$ has derivatives everywhere and $f^{(n)}(0) = 0$ for all n.
But observe that the Taylor series of $f(x)$ centered at 0 is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = \sum_{n=0}^{\infty} 0 = 0.$$

Because $T(x) = 0$, $T(x)$ converges for all values of x and the interval of convergence is all real numbers $(-\infty, \infty)$.
Does this mean $f(x) = T(x) = 0$ for all real numbers x?

No. $f(x)$ is an example of a function that is **not equal** to its Taylor series inside the interval of convergence.

8.6 When is a function equal to its Taylor series?

To make sure that a function $f(x)$ can be approximated by its Taylor series $T(x)$, we need to compute the **difference** between $f(x)$ and $T(x)$.

Recall the definition of the k**th-degree Taylor polynomial of** $f(x)$ **centered at** a:

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

We defined the **Taylor series** as the limit of the sequence of Taylor polynomials:

$$T(x) = \lim_{k\to\infty} T_k(x)$$

8.6 When is a function equal to its Taylor series?

We say that $f(x)$ is **equal** to its Taylor series if the sequence of Taylor polynomials $T_k(x)$ **converges** to $f(x)$:

$$f(x) = \lim_{k\to\infty} T_k(x)$$

We define $R_k(x) = f(x) - T_k(x)$ as the kth degree **remainder** (or the **error**) of the Taylor series. Then $f(x)$ is equal to its Taylor series if and only if the **remainder (error) vanishes**, i.e.

$$\lim_{k\to\infty} R_k(x) = 0$$

Taylor's Inequality

Suppose $T_k(x)$ is a Taylor polynomial centered at a for the function f. Let d be a constant and $|f^{(k+1)}(x)| \leq M$ for values of x satisfying $|x-a| \leq d$. Then for those values of x, the error $R_k(x)$ of the Taylor polynomial $T_k(x)$ satisfies the inequality

$$|R_k(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1} \leq \frac{M}{(k+1)!}d^{k+1}$$

In other words, the error from $T_k(x)$ is bounded by some constants;

$$|R_k(x)| \leq \frac{M}{(k+1)!}d^{k+1}$$

Deciphering Taylor's Inequality:

1. $|x - a| \le d$ looks **very similar** to the inequality $|x - a| < R$ (R is the radius of convergence.)

2. a is the **center** of the Taylor polynomial, and it is the center of the intervals.

3. d is the **radius of approximation**, which is the distance from the center to the boundary of the **interval of approximation**. In order for the approximation to make sense, d must be less than R:

$$d < R.$$

4. M is computed by **maximizing** $|f^{(k+1)}(x)|$ in the interval of approximation $[a - d, a + d]$. (Usually maximizing an increasing, decreasing, or an oscillating function. Techniques like the Closed Interval Method can be used.)

Controlling the Error

There are three moving parts to Taylor's Inequality:

1. k, the degree of the Taylor polynomial
2. d, the radius of approximation.
3. M, the maximum bound for the $(k+1)$-th derivative of $f(x)$ inside the interval of approximation.

The last moving part M **is dependent on both** k **and** d since the maximum of the $(k+1)$-th derivative is taken over the interval $[a-d, a+d]$.

The **error gets smaller** $(|R_k| \to 0)$ as one either

1. Increases the degree k of the Taylor polynomial $(k \to \infty)$ or
2. Reduces the size of the interval of approximation $(d \to 0)$.

Desmos Examples to Play With

Taylor Polynomials of degree k and the radius of approximation d:

https://www.desmos.com/calculator/bdhuwxcgm7

Graphs of the Taylor polynomials and the errors for various functions:

https://www.cengage.com/math/discipline_content/stewartccc4/2008/14_cengage_tec/publish/deployments/concepts_4e/4c3_tool.html#