Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

List of power series (centered at 0) that you must memorize. "I" means Interval of Convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad \text{I: } (-1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{I: } (-\infty, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{I: } (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{I: } (-\infty, \infty)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{I: } [-1, 1]$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{I: } (-1, 1]$$

8.7 Taylor Series

Compute the Taylor series of $f(x) = \dfrac{1}{1-x}$ centered at 0 and find its interval of convergence.

$f(x) = \dfrac{1}{1-x}$

$f'(x) = \dfrac{1}{(1-x)^2}$

$f''(x) = \dfrac{2}{(1-x)^3}$

$f^{(3)}(x) = \dfrac{3 \cdot 2}{(1-x)^4}$

$f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}}$

$f^{(n)}(0) = n!$

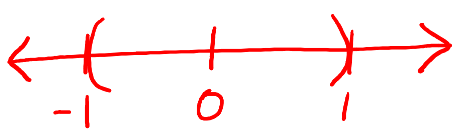
$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x = \sum_{n=0}^{\infty} x^n$$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty} |x| = |x|.$$

By the Ratio Test, $T(x)$ converges absolutely when $|x| < 1$.

possible interval of convergence



check the boundary points

$x=-1$, $x=1$ for convergence

$x=-1$

$$\sum_{n=0}^{\infty} (-1)^n \qquad \lim_{n\to\infty} |(-1)^n| = 1 \neq 0$$

$x=1$

$$\sum_{n=0}^{\infty} 1^n \qquad \lim_{n\to\infty} 1^n = 1 \neq 0$$

Both diverge by the Divergence Test.

Therefore $(-1, 1)$ is the interval of convergence

Find the 42nd derivative of $\sin(x^2)$ at $x = 0$. $\leftarrow$ Asking for $f^{(42)}(0)$.

Recall that the Taylor series of a function centered at a encodes all of the derivatives evaluated at a. $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

The Taylor series for $\sin x$ centered at 0 is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Then since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$,

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}.$$

Note that the general formula for a Taylor series centered at 0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(42)}(0)}{42!}x^{42} + \cdots$$

Observe that the 42nd degree term has $f^{(42)}(0)$ in its coefficient.

This means that the 42nd degree term of the power series

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \quad \text{has} \quad f^{(42)}(0).$$

Solving for n,

$$x^{4n+2} = x^{42}$$

$$4n+2 = 42$$

$$n = 10$$

Hence we have

$$\frac{(-1)^{10} x^{42}}{21!} = \frac{f^{(42)}(0)\, x^{42}}{42!}.$$

Solving for $f^{(42)}(0)$,

$$\frac{1}{21!} = \frac{f^{(42)}(0)}{42!}, \qquad f^{(42)}(0) = \frac{42!}{21!}.$$

Euler's Formula

Find the Maclaurin series of the function e^{ix} where $i = \sqrt{-1}$ is the imaginary unit.

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \underbrace{\sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}}_{\text{Split into even and odd}}$

$$= \sum_{n=0}^{\infty} \frac{i^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1} x^{2n+1}}{(2n+1)!}$$

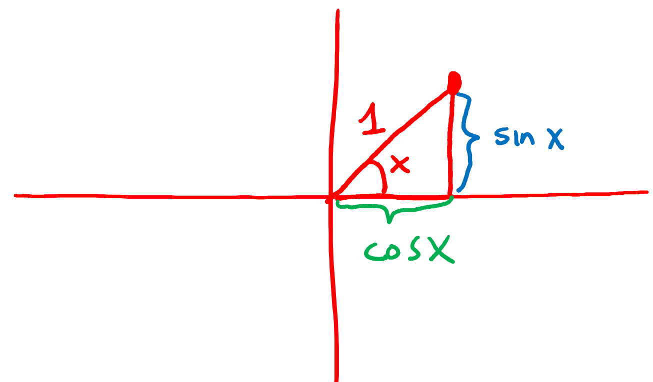
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n i \, x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \cos x + i \sin x$$

In conclusion,

$$e^{ix} = \cos x + i \sin x$$



Find the sum of the series $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$

Write the series in sigma notation.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}$$

Recall: $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

If we plug in $x = \frac{1}{2}$ to the above equality, we get

$$\ln\left(1+\frac{1}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}.$$

Therefore

$$\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n} = \ln\left(1+\frac{1}{2}\right) = \ln\left(\frac{3}{2}\right).$$

Find the sum of the series $\dfrac{1}{1\cdot 2} - \dfrac{1}{3\cdot 2^3} + \dfrac{1}{5\cdot 2^5} - \dfrac{1}{7\cdot 2^7} + \cdots$

series in sigma notation:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)\,2^{2n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1}$$

Recall: $\arctan x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

If we plug in $x = \frac{1}{2}$ to the above equality, we get

$$\arctan\left(\tfrac{1}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1}.$$

Therefore

$$\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1} = \arctan\left(\tfrac{1}{2}\right).$$

8.7 Taylor Series

The **Taylor series of** $f(x)$ **centered at** a is defined as

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Dissecting the Notations

n th derivative
of f(x) evaluated
at a.

center of a Taylor Series

Taylor Series $\rightarrow$ $$T(x) = \boxed{\sum_{n=0}^{\infty}} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

n iterates

n = 0, 1, 2, ...

Find the Taylor series for $f(x) = x^4 - 3x^2 + 1$ centered at $a = 0$.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f^{(0)}(x) = f(x) = x^4 - 3x^2 + 1$

$f^{(1)}(x) = 4x^3 - 6x$

$f^{(2)}(x) = 12x^2 - 6$

$f^{(3)}(x) = 24x$

$f^{(4)}(x) = 24$

$f^{(5)}(x) = 0$

$f^{(6)}(x) = 0$

$\vdots$

$f(0) = 1$

$f^{(1)}(0) = 0$

$f^{(2)}(0) = -6$

$f^{(3)}(0) = 0$

$f^{(4)}(0) = 24$

$f^{(5)}(0) = 0$

$f^{(6)}(0) = 0$

$\vdots$

Then

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5 + \cdots$$

$$= 1 + 0 \cdot x + \frac{-6}{2}x^2 + 0 \cdot x^3 + \frac{24}{24}x^4 + 0 \cdot x^5 + 0 \cdot x^6 + \cdots$$

$$= 1 - 3x^2 + x^4$$

$$= x^4 - 3x^2 + 1$$

Why does the Taylor series look the same as the original function?

It's because the function was already a polynomial centered at 0. A Taylor series is a polynomial approximation so nothing happened as a result.

Find the Taylor series for $f(x) = x^4 - 3x^2 + 1$ centered at $a = 1$.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$f(x) = x^4 - 3x^2 + 1$ $\quad f(1) = 1 - 3 + 1 = -1$

$f'(x) = 4x^3 - 6x$ $\quad f'(1) = 4 - 6 = -2$

$f''(x) = 12x^2 - 6$ $\quad f''(1) = 12 - 6 = 6$

$f^{(3)}(x) = 24x$ $\quad f^{(3)}(1) = 24$

$f^{(4)}(x) = 24$ $\quad f^{(4)}(1) = 24$

$f^{(5)}(x) = 0$ $\quad f^{(5)}(1) = 0$

$f^{(6)}(x) = 0$ $\quad f^{(6)}(1) = 0$

$\vdots \quad \vdots$

$$T(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 + \cdots$$

$$= -1 + (-2)(x-1) + \frac{6}{2}(x-1)^2 + \frac{24}{6}(x-1)^3 + \frac{24}{24}(x-1)^4 + \frac{0}{5!}(x-1)^5 + \frac{0}{6!}(x-1)^6 + \cdots$$

$$T(x) = -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

8.7 Taylor Series

$a = \frac{\pi}{3}$

Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\frac{\pi}{3}$.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{3}\right)}{n!}\left(x-\frac{\pi}{3}\right)^n$$

Compute the derivatives

$f = \sin x \qquad f^{(4)} = \sin x \qquad f^{(8)} = \sin x$

$f' = \cos x \qquad f^{(5)} = \cos x \qquad \vdots$

$f'' = -\sin x \qquad f^{(6)} = -\sin x$

$f''' = -\cos x \qquad f^{(7)} = -\cos x$

$f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad f^{(4)}\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad f^{(5)}\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f''\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad f^{(6)}\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f'''\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2} \qquad f^{(7)}\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$\vdots$

$$T(x) = f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x-\frac{\pi}{3}\right)^1 + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \cdots$$

$$= \frac{\sqrt{3}}{2} + \frac{\frac{1}{2}}{1!}\left(x-\frac{\pi}{3}\right)^1 + \frac{-\frac{\sqrt{3}}{2}}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{-\frac{1}{2}}{3!}\left(x-\frac{\pi}{3}\right)^3 + \frac{\frac{\sqrt{3}}{2}}{4!}\left(x-\frac{\pi}{3}\right)^4 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{3}}{2} \frac{(-1)^n \left(x-\frac{\pi}{3}\right)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{1}{2} \frac{(-1)^n \left(x-\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$$

Desmos Examples to Play With

Assortment of functions and their Taylor series:

https://www.desmos.com/calculator/dupl0xq1ke

Taylor Series of $\sin(x)$ with different centers:

https://www.desmos.com/calculator/dghngmfvoi