Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

7.4 Exponential Growth and Decay

For modeling **uninhibited population growth** or **natural decay**, we use

$$\frac{dP}{dt} = kP$$

where k is a constant. If $k > 0$, the differential equation models growth and if $k < 0$, it models natural decay.

The quantity $\dfrac{1}{P}\dfrac{dP}{dt}$ is called **the relative growth rate.**

To solve the differential equation $\frac{dP}{dt} = kP$, observe that it is separable.

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{dP}{P} &= k\ dt \\
\int \frac{dP}{P} &= \int k\ dt \\
\ln|P| &= kt + C \\
|P| &= e^{kt+C} \\
P &= \pm e^{C} e^{kt} \\
P &= Ae^{kt}
\end{aligned}$$

where A is a constant. If our population has initial value $P(0) = P_0$, then

$$\begin{aligned}
P_0 &= Ae^{0} \\
P_0 &= A
\end{aligned}$$

Hence $P = P_0 e^{kt}$ is the solution to the exponential growth and decay problem.

7.4 Radioactive Decay

Recall that **Half-life** is the time required for half of any given quantity to decay.
The half-life of radium-226 is 1590 years.

1. A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of ${}^{226}_{88}\mathrm{Ra}$ that remains after t years.

2. Find the mass after 1000 years correct to the nearest milligram.

3. When will the mass be reduced to 30 mg?

1. Assuming that radioactive decay is an exponential decay problem, the solution is given by

$$M(t) = M_0 e^{kt}$$

for some constant k. Since the initial mass of radium is 100 mg, let $M_0 = 100$. To find k, we use the half-life of 1590 years.

$$\begin{aligned}
50 &= 100e^{k \cdot 1590} \\
\frac{1}{2} &= e^{1590k} \\
\ln\left(\frac{1}{2}\right) &= 1590k \\
k &= \frac{\ln\left(\frac{1}{2}\right)}{1590} \\
k &\approx -0.00043594
\end{aligned}$$

Hence $M(t) = 100e^{\frac{\ln(1/2)}{1590}t}$ where M is in milligrams and t is in years.

2. Plug in 1000 for t to obtain

$$M(1000) = 100e^{\frac{\ln(1/2)}{1590}\cdot 1000}$$
$$M(1000) \approx 64.6655 \text{ mg}$$

Hence the mass after 1000 years is approximately 65 milligram.

3. Plug in 30 for M and solve for t.

$$30 = 100e^{\frac{\ln(1/2)}{1590}t}$$

$$\frac{3}{10} = e^{\frac{\ln(1/2)}{1590}t}$$

$$\ln(3/10) = \frac{\ln(1/2)}{1590}t$$

$$t = \frac{1590\ln(3/10)}{\ln(1/2)}$$

$$t \approx 2761.775$$

It will take approximately 2762 years for the mass to be reduced to 30 mg.

7.4 Newton's Law of Cooling

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings (assumed to be constant), then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant.

7.4 Newton's Law of Cooling

Newtons Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Note that this law also applies to warming.

Since the given differential equation is separable, we can apply the usual integration technique to obtain a general solution.

7.4 Newton's Law of Cooling

A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

1. What is the temperature of the soda pop after another half hour?
2. How long does it take for the soda pop to cool to 50°F?

1. To make computation easier for solving the differential equation $\frac{dT}{dt} = k(T - T_s)$, we can let $y = T - T_s$ so that

$$\frac{dy}{dt} = \frac{dT}{dt} = k(T - T_s) = ky$$

and use the general solution $y(t) = y_0 e^{kt}$ for the exponential growth and decay problem. Then $y = T - 44$ and $y_0 = 72 - 44 = 28$ so

$$y(t) = y_0 e^{kt} = 28e^{kt}.$$

Since the soda has cooled to 61°F after 30 minutes, $T(30) = 61$ and

$$y(30) = T(30) - 44$$
$$y(30) = 61 - 44 = 17$$

Using this, we can solve for k:

$$\begin{aligned} y(30) &= 28e^{k30} \\ 17 &= 28e^{30k} \\ \frac{17}{28} &= e^{30k} \\ \ln(17/28) &= 30k \\ k &= \frac{\ln(17/28)}{30} \\ k &\approx -0.01663 \end{aligned}$$

Thus

$$\begin{aligned} y(t) &= 28e^{-0.01663t} \\ T(t) - 44 &= 28e^{-0.01663t} \\ T(t) &= 44 + 28e^{-0.01663t}. \end{aligned}$$

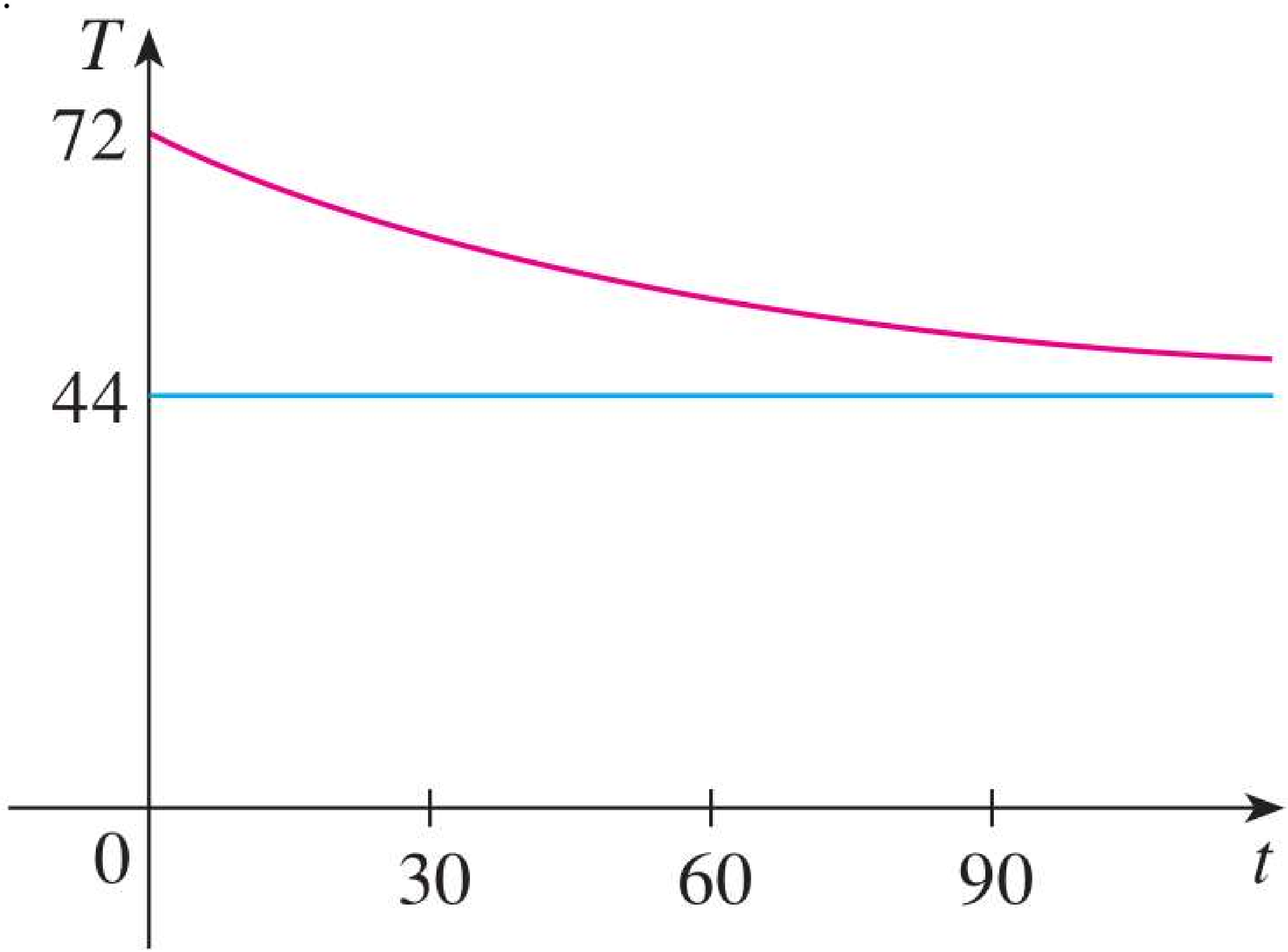
Therefore after another half hour, $t = 60$ and the soda has cooled to $T(60) = 44 + 28e^{-0.01663 \cdot 60} = 54.32$ degrees Fahrenheit.

2. Set the temperature to 50 degrees and solve for time.

$$\begin{aligned} T(t) &= 44 + 28e^{-0.01663t} \\ 50 &= 44 + 28e^{-0.01663t} \\ 6 &= 28e^{-0.01663t} \\ \frac{6}{28} &= e^{-0.01663t} \\ \ln(6/28) &= -0.01663t \\ t &= \frac{\ln(6/28)}{-0.01663} \\ t &\approx 92.63 \end{aligned}$$

It takes the soda 92.63 minutes to cool to 50°F.

The graph of $T(t) = 44 + 28e^{-0.01663t}$ (red) and its limiting value (blue) as $t \to \infty$.



7.4 Compounded Interest

Suppose we have \$1000 invested with 6% annual interest.

What is our net worth after 2 years given that interest is paid

1. Annually
2. Quarterly
3. Monthly
4. n times a year

1. On year 0, we have \$1000. After 1 year, we have

$$1000 \cdot (1.06) = 1060 \text{ dollars.}$$

After 2 years, we have

$$1060 \cdot (1.06) = 1123.6 \text{ dollars.}$$

Note that the formula is

$$y(t) = 1000(1.06)^t$$

where t is in years.

2. Quarterly means every three months or 4 times a year. Since the interest rate is 6% annually, we have to divide 6% by 4 to get the interest rate for each quarter. In other words, the interest rate is $6/4 = 1.5$ percent per quarter. Initially, we have \$1000. After 1 quarter, we have

$$1000 \cdot (1.015) = 1015 \text{ dollars.}$$

After 2 quarters, we have

$$1015 \cdot (1.015) = 1030.225 \text{ dollars.}$$

After q quarters, we have

$$1000(1.015)^q \text{ dollars.}$$

Therefore since there are 8 quarters in two years, our networth becomes

$$1000(1.015)^8 = 1126.49 \text{ dollars.}$$

Note that we can change the time variable to be in t years instead of quarters by letting $q = 4t$.

$$y(t) = 1000(1.015)^{4t}$$

3. Monthly. Since the interest rate is 6% annually, we have to divide 6% by 12 to get the interest rate for each month. In other words, the interest rate is $6/12 = 0.5$ percent per quarter.
Initially, we have \$1000. After 1 month, we have

$$1000 \cdot (1.005) = 1005 \text{ dollars.}$$

After 2 months, we have

$$1005 \cdot (1.005) = 1010.025 \text{ dollars.}$$

After m months, we have

$$1000(1.005)^m \text{ dollars.}$$

Therefore since there are 24 months in two years, our networth becomes

$$1000(1.005)^{24} = 1127.16 \text{ dollars.}$$

Note that we can change the time variable to be in t years instead of months by letting $m = 12t$.

$$y(t) = 1000(1.005)^{12t}$$

4. n times a year. Since the interest rate is 6% annually, we have to divide 6% by n to get the interest rate for each compounding period. Since there are nt compounding periods in t years, our networth after t years is given by

$$y(t) = 1000\left(1 + \frac{0.06}{n}\right)^{nt}$$

Now if we let $n \to \infty$, we will be compounding the interest **continuously** and our networth will be in t years

$$\begin{aligned} y(t) &= \lim_{n\to\infty} 1000\left(1 + \frac{0.06}{n}\right)^{nt} \\ &= \lim_{n\to\infty} 1000\left(1 + \frac{1}{n/0.06}\right)^{(n/0.06)0.06t} \\ &= 1000\left[\lim_{n\to\infty}\left(1 + \frac{1}{n/0.06}\right)^{(n/0.06)}\right]^{0.06t} \end{aligned}$$

Letting $m = n/0.06$, we see that $m \to \infty$ as $n \to \infty$. Then since $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$,

$$\begin{aligned} y(t) &= 1000\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^{0.06t} \\ &= 1000e^{0.06t} \end{aligned}$$

Therefore with continuously compounding interest at the rate of 6% per year, the networth after t years is

$$y(t) = 1000e^{0.06t}.$$

In general, with the annual rate of r and the initial amount y_0, the networth after t years of continuously compounding interest is

$$y(t) = y_0 e^{rt}.$$