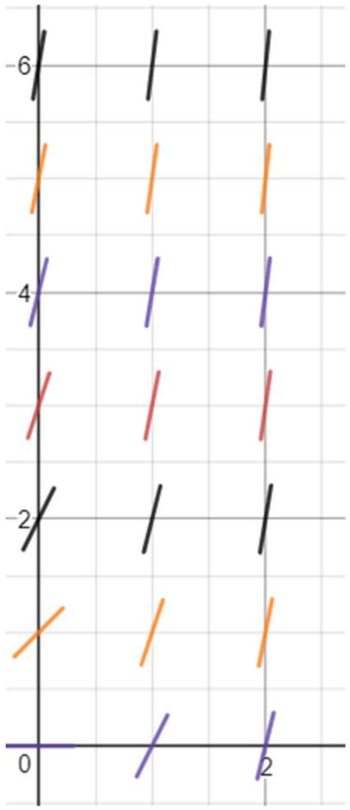
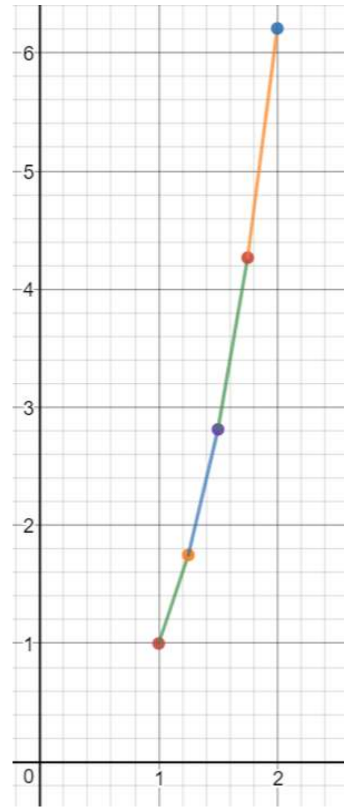
7.3 Separable Equations

We use Euler's Method on the slope fields to approximate the solutions to differential equations:

Slope field



Euler's method



7.3 Separable Equations

In some cases, we can explicitly solve the differential equations.

A **separable equation** is a first-order differential equation in which the expression for $\dfrac{dy}{dx}$ can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

7.3 Implicit and Explicit Solutions

A solution to a differential equation is called **explicit** if y is isolated in the equation.

On the other hand, a solution is **implicit** if y is not isolated.

Example:

$$y = \sqrt{4 - x^2} \quad \text{(Explicit)}$$

$$x^2 + y^2 = 4 \quad \text{(Implicit)}$$

EXAMPLE 1 **Solving a separable equation**

(a) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

(a) $\frac{dy}{dx} = \frac{x^2}{y^2}$

move all the y's to the left and move all the x's to the right.

$y^2 dy = x^2 dx$

Integrate both sides

$\int y^2 dy = \int x^2 dx$

$\frac{y^3}{3} = \frac{x^3}{3} + c$ (implicit solution)

$y^3 = x^3 + 3c$

$y = \sqrt[3]{x^3+3c}$ (explicit solution)

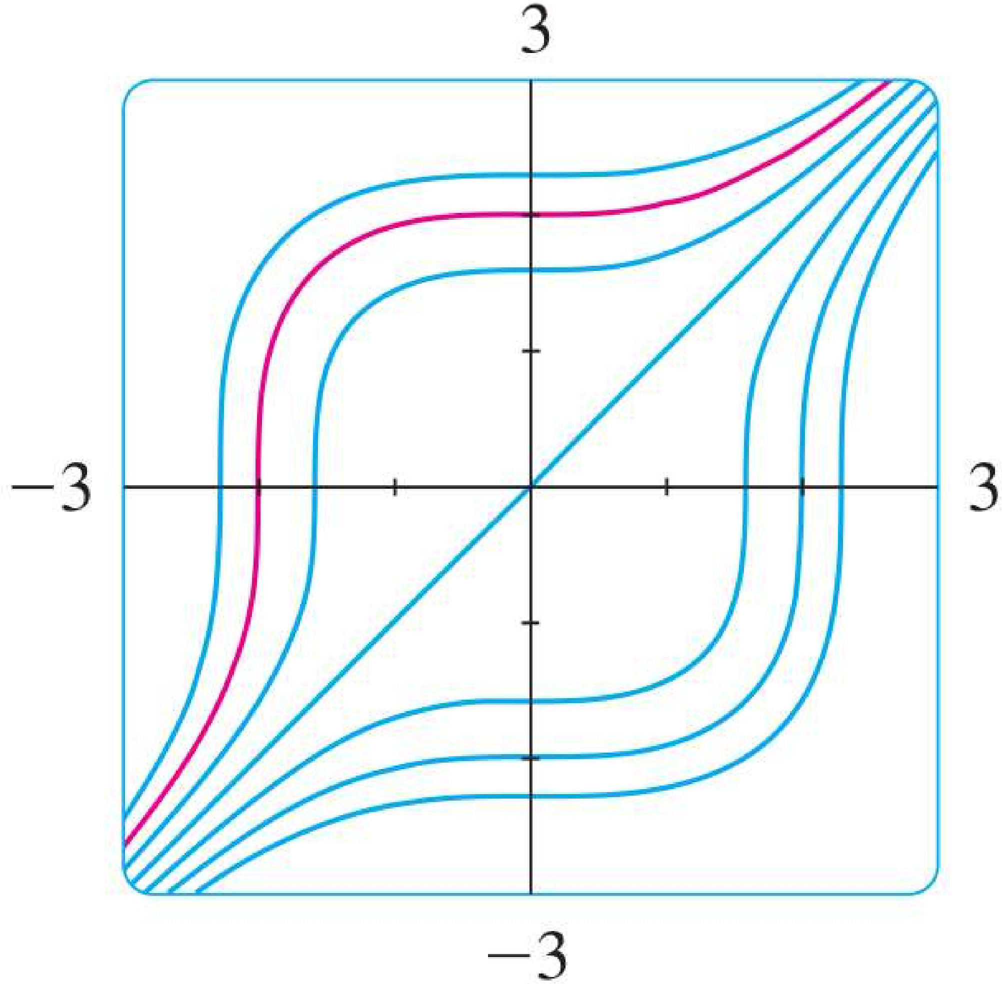
(b) Implicit solutions have the most information so we plug in the initial value $y(0)=2$ on $\frac{y^3}{3}=\frac{x^3}{3}+c$ to obtain

$$\frac{8}{3}=\frac{0}{3}+C$$

$$\frac{8}{3}=C$$

So $\frac{y^3}{3}=\frac{x^3}{3}+\frac{8}{3}$ is the implicit solution of $\frac{dy}{dx}=\frac{x^2}{y^2}$ with the initial value $y(0)=2$.

Solutions to $\frac{dy}{dx} = \frac{x^2}{y^2}$. The solution for the initial value $y(0) = 2$ is in red.



EXAMPLE 2 A separable equation with an implicit solution

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

$$\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$$

$$(2y + \cos y)\,dy = 6x^2\,dx$$

$$\int (2y+\cos y)\,dy = \int 6x^2\,dx$$

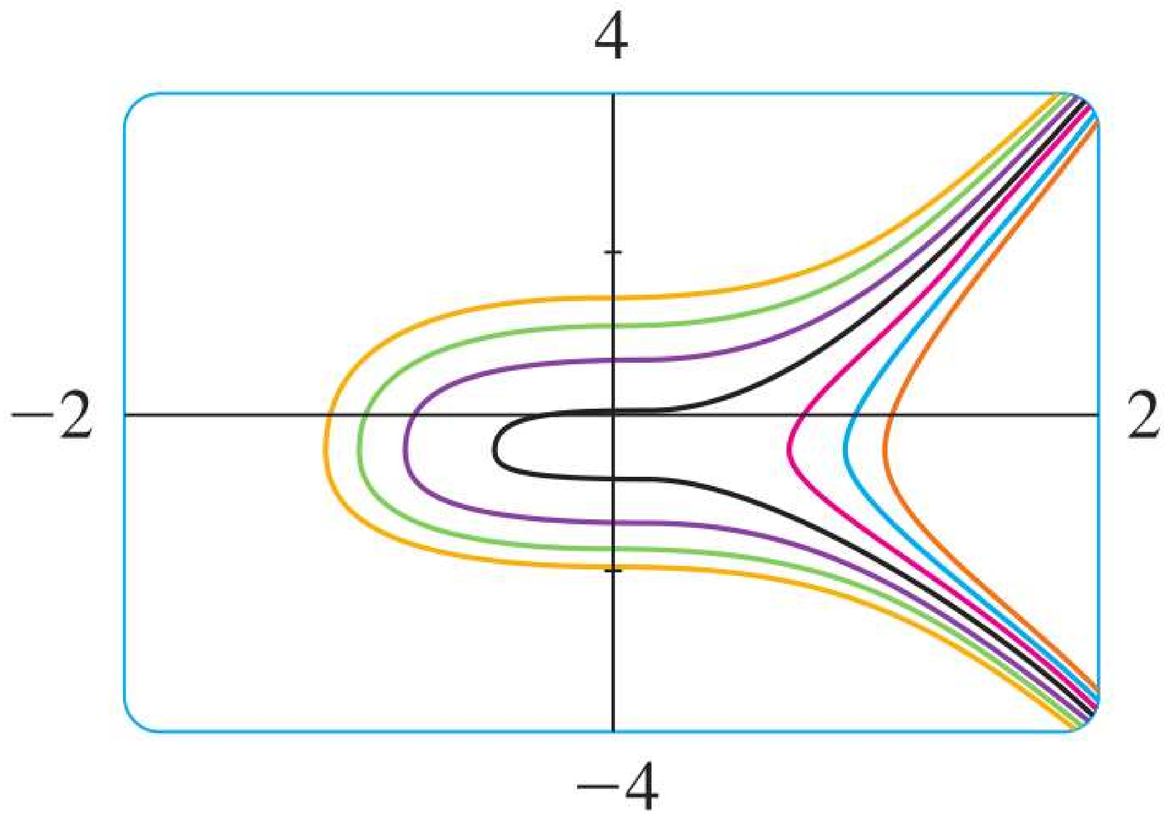
$$y^2 + \sin y = 2x^3 + C$$

we can't solve for y so there's no explicit solution.

Since no initial value was given, $y^2 + \sin y = 2x^3 + C$ is the general implicit solution to the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y+\cos y}$.

Solutions to the differential equation

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$



EXAMPLE 3 Solve the equation $y' = x^2 y$.

Since $y' = \frac{dy}{dx}$, we have

$\frac{dy}{dx} = x^2 y$. Then

$$\frac{dy}{y} = x^2 dx$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

$$e^{\ln|y|} = e^{\frac{x^3}{3} + C}$$

$$|y| = e^C e^{x^3/3}$$

$$y = \pm e^C e^{x^3/3}.$$

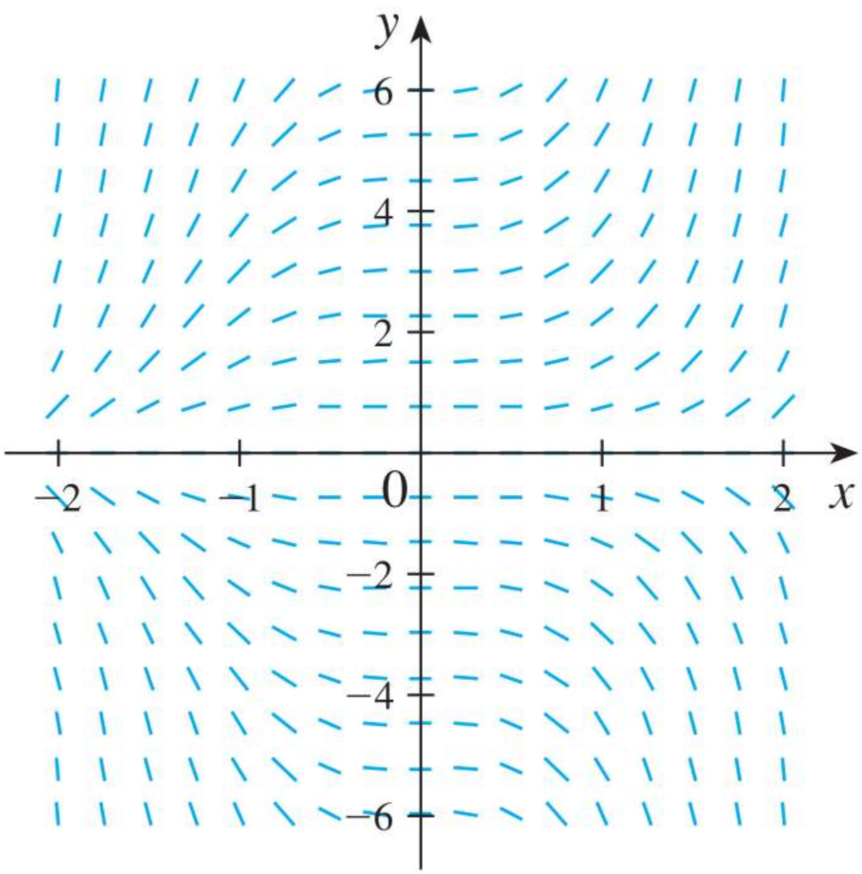
Note that $\pm e^C$ can be any nonzero real number.

Observe that $y = 0$ is an equilibrium solution since it satisfies the diff eq $y' = x^2 y$.

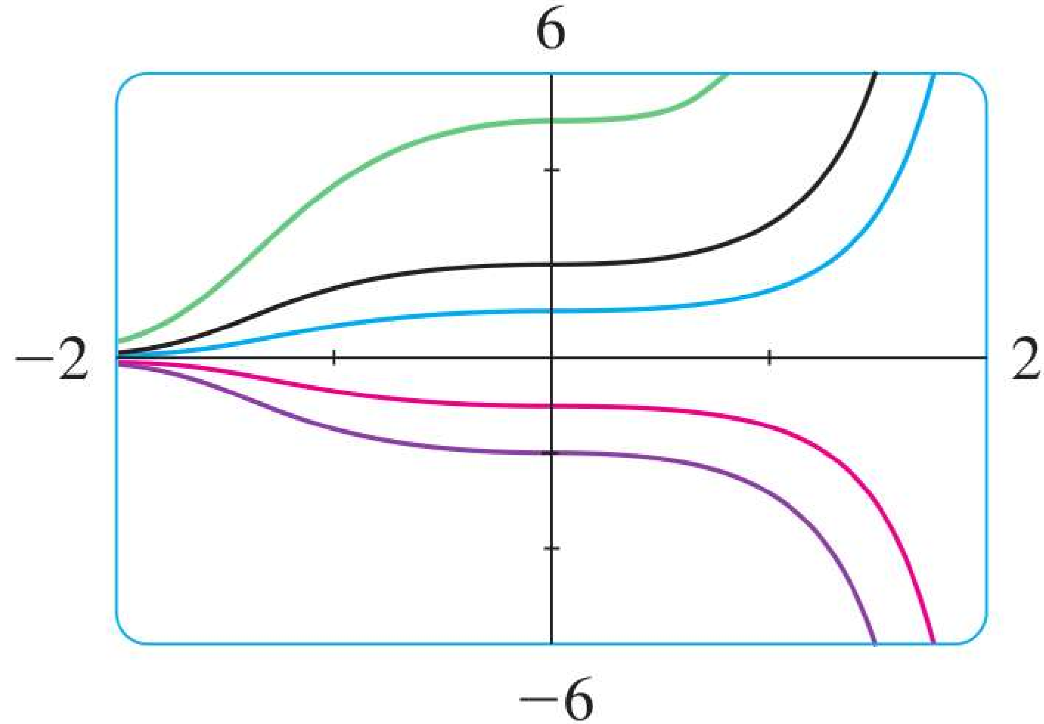
Therefore

$y = A e^{x^3/3}$ where A is any real number is the general explicit solution to the diff eq $y' = x^2 y$

Slope field for $y' = x^2y$



Solutions for $y' = x^2y$

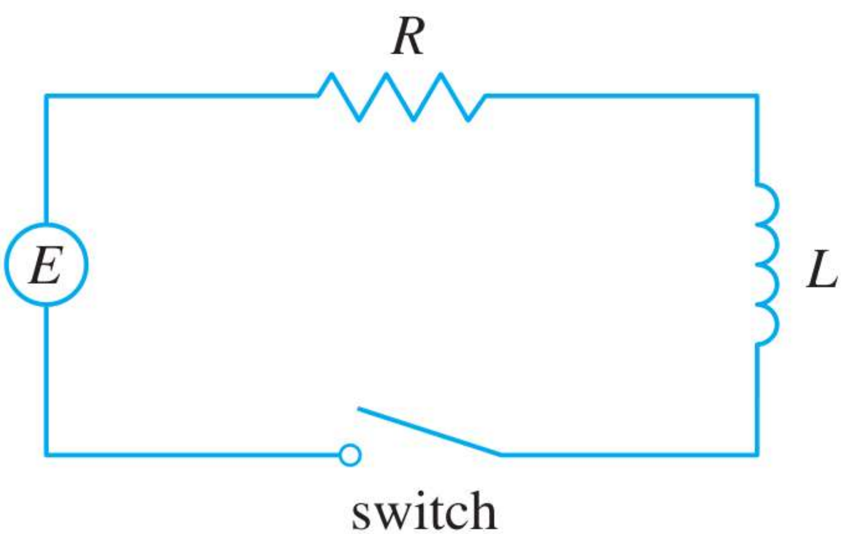


The current in the electric circuit shown below is modeled by the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

1. Find an expression for the current in the circuit when the resistance is 12 Ω, the inductance is 4 H, the battery gives a constant voltage of 60 V, and the switch is turned on when $t = 0$.

2. What is the limiting value of the current?



① $R=12$, $L=4$, $E=60$

$$L\frac{dI}{dt}+RI=E(t)$$

$$4\frac{dI}{dt}+12I=60$$

$$\frac{dI}{dt}=15-3I$$

Solving for I,

$$\frac{dI}{15-3I}=dt$$

$$\int\frac{dI}{15-3I}=\int dt$$

$$\frac{1}{-3}\ln|15-3I|=t+C$$

$$\ln|15-3I|=-3t-3C$$

The switch is turned on when $t=0$ means the current is 0 at $t=0$, i.e. $I(0)=0$. This is the initial value for the current.

Continuing to solve for I,

$$e^{\ln|15-3I|}=e^{-3t-3C}$$

$$|15-3I|=e^{-3C}e^{-3t}$$

$$15-3I=\pm e^{-3C}e^{-3t}$$

$$I=5\pm\frac{e^{-3C}}{3}e^{-3t}.$$

Let $A=\pm\frac{e^{-3C}}{3}$. Then A can be any real number.

Then $I = 5 + Ae^{-3t}$. Use the initial value $I(0)=0$.

$0 = 5 + Ae^{0}$

$0 = 5 + A$

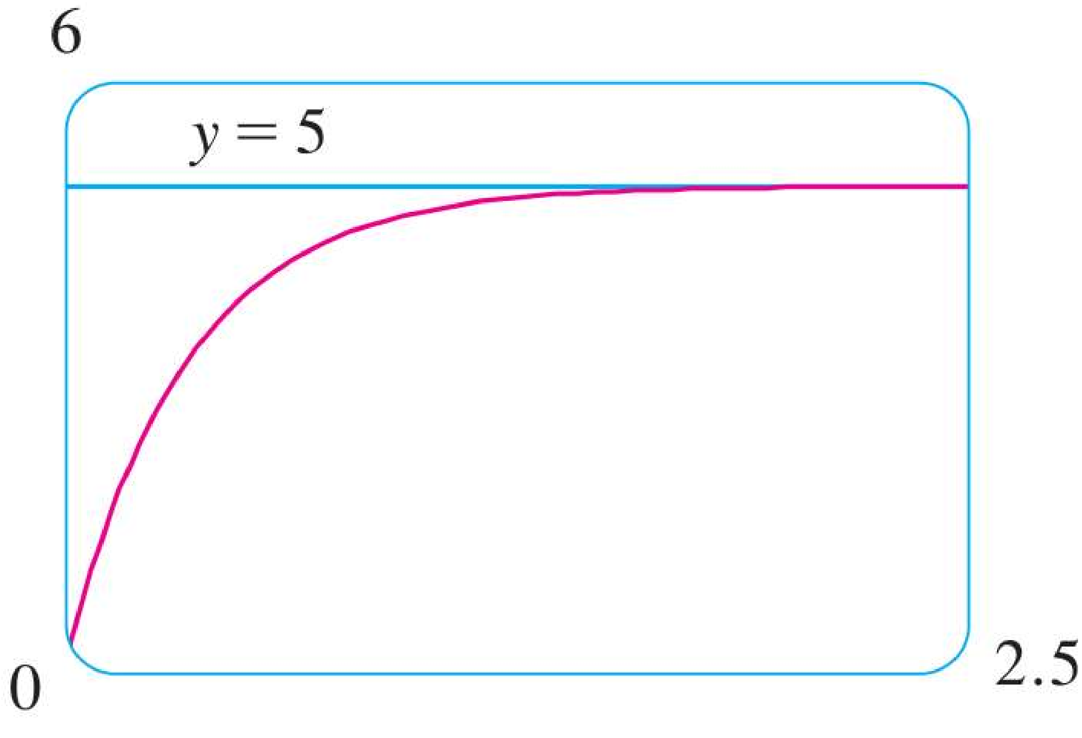
$A = -5$

Hence $I = 5 - 5e^{-3t}$

② The limiting value of the current is

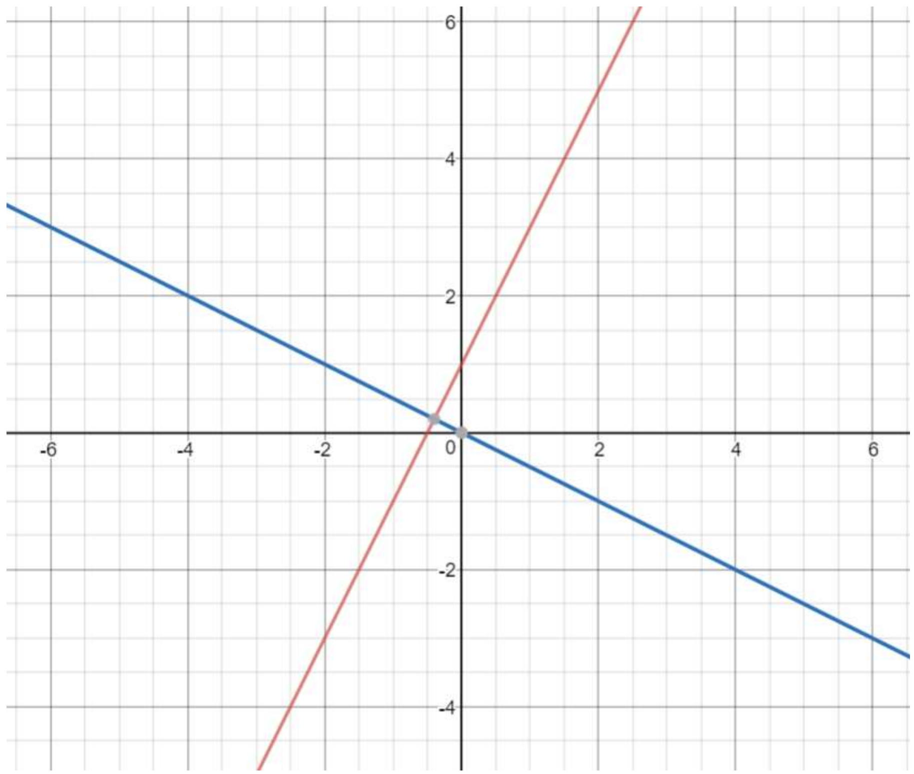
$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} 5 - 5e^{-3t} = 5 - 0 = 5 \text{ amp}$$

The graph of $I(t)$ and its limiting value.



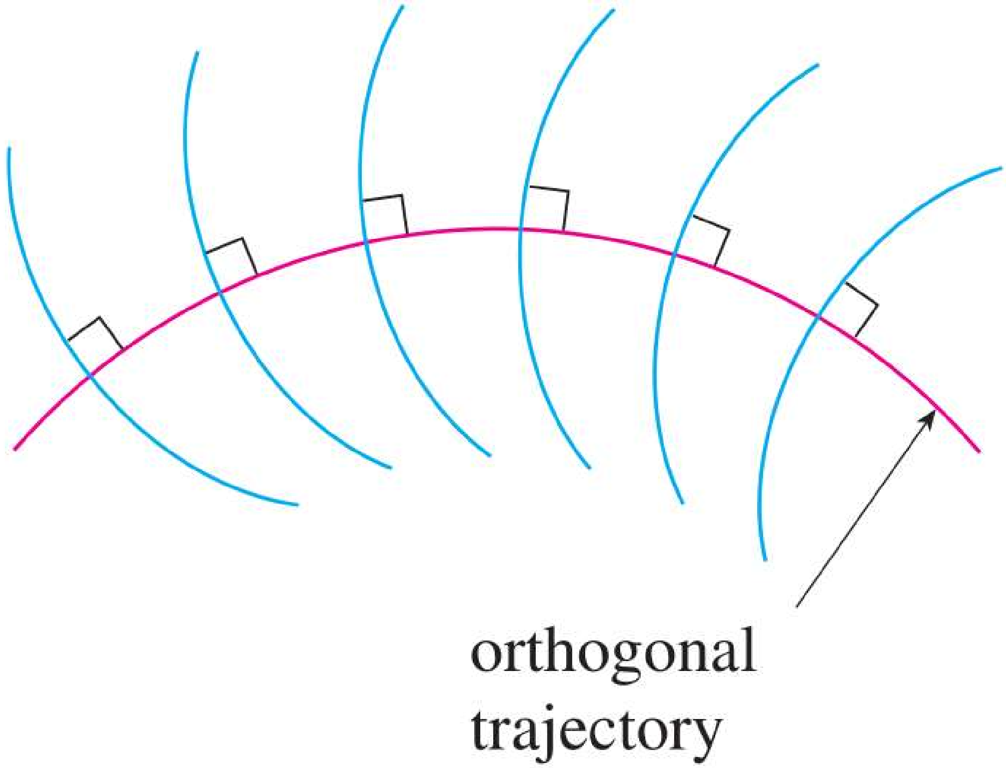
7.3 Orthogonal (Perpendicular) Lines

Recall: Two lines $y = m_1x + b_1$ and $y = m_2x + b_2$ are penpendicular if $m_1m_2 = -1$.

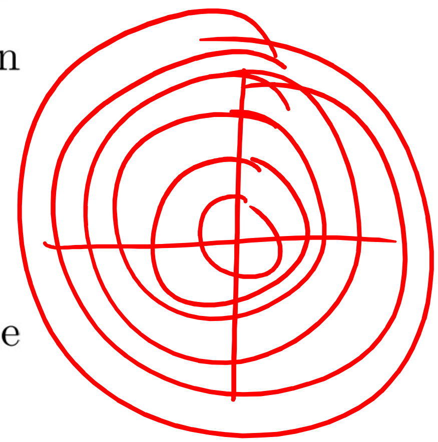


7.3 Orthogonal Trajectories

An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally (perpendicularly).



Since a differential equation describes a famliy of curves, an orthogonal trajectory can be computed by multiplying $\frac{dy}{dx}$ together and setting the expression equal to -1.



Example: The family of concentric circles at the origin has the formula $x^2 + y^2 = r^2$ where r is any real number.
The first order differential equation that describes the concentric circles can be obtained by implicitly differentiating the equation $x^2 + y^2 = r^2$.

$$2x\ dx + 2y\ dy = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

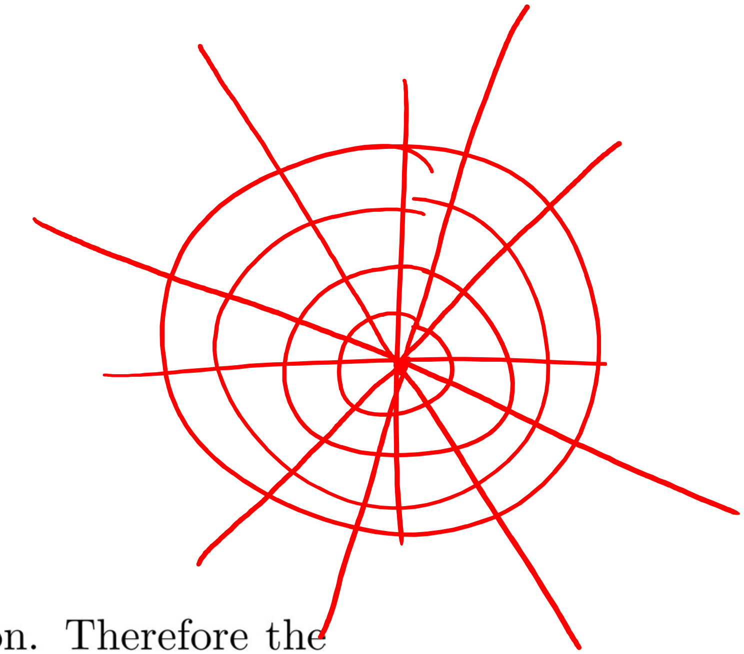
Then the differential equation that describes the orthogonal trajectory to the circles is

$$\left(-\frac{x}{y}\right)\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{y}{x}$$

Solving this differential equation gets us

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{dy}{y} &= \frac{dx}{x} \\
\ln|y| &= \ln|x| + C \\
|y| &= e^C|x| \\
y &= \pm e^C x \\
y &= Ax
\end{aligned}$$



where A can be any real number since $y = 0$ is also a solution. Therefore the orthogonal trajectories of the concentric circles at the origin are the straight lines through the origin.

7.3 Orthogonal Trajectories

Family of concentric circles centered at the origin:

$x^2 + y^2 = r^2$

Differential equation:

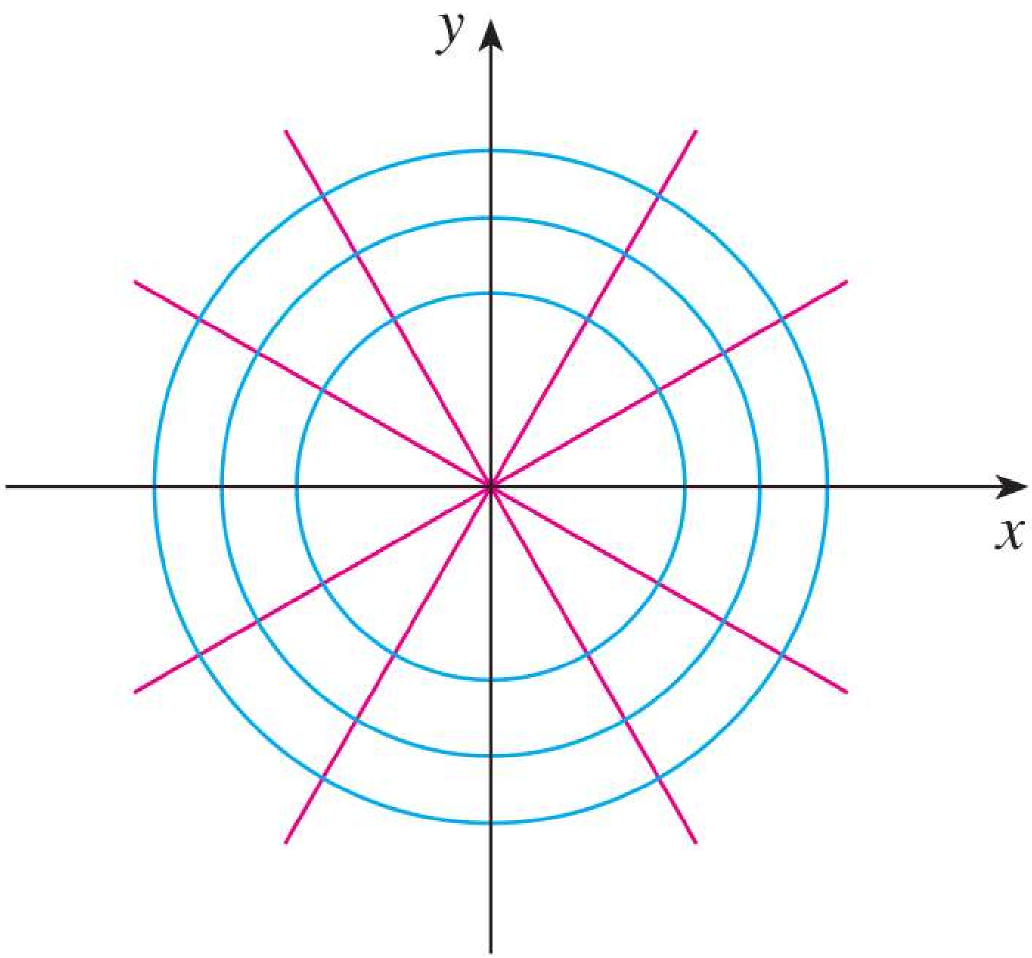
$$\frac{dy}{dx} = -\frac{x}{y}$$

Family of straight lines through the origin:

$y = mx$

Differential equation:

$$\frac{dy}{dx} = \frac{y}{x} = \frac{mx}{x} = m$$



Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitrary constant.

Solution. First implicitly differentiate $x = ky^2$ and solve for $\frac{dy}{dx}$.

$$dx = k \cdot 2y\ dy$$

$$\frac{dy}{dx} = \frac{1}{2ky}$$

Now observe that k is not entirely independent of the variables x and y. Because $x = ky^2$, we must have that $k = \frac{x}{y^2}$. Substituting k in the above equation, we get

$$\frac{dy}{dx} = \frac{1}{2\left(\frac{x}{y^2}\right)y} = \frac{y}{2x}$$

Now the derivative for the orthogonal trajectory must be the reciprocal of the above expression times -1. In other words,

$$\left(\frac{y}{2x}\right)\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{2x}{y}$$

Solving the orthogonal trajectory's differential equation,

$$\frac{dy}{dx} = \frac{-2x}{y}$$

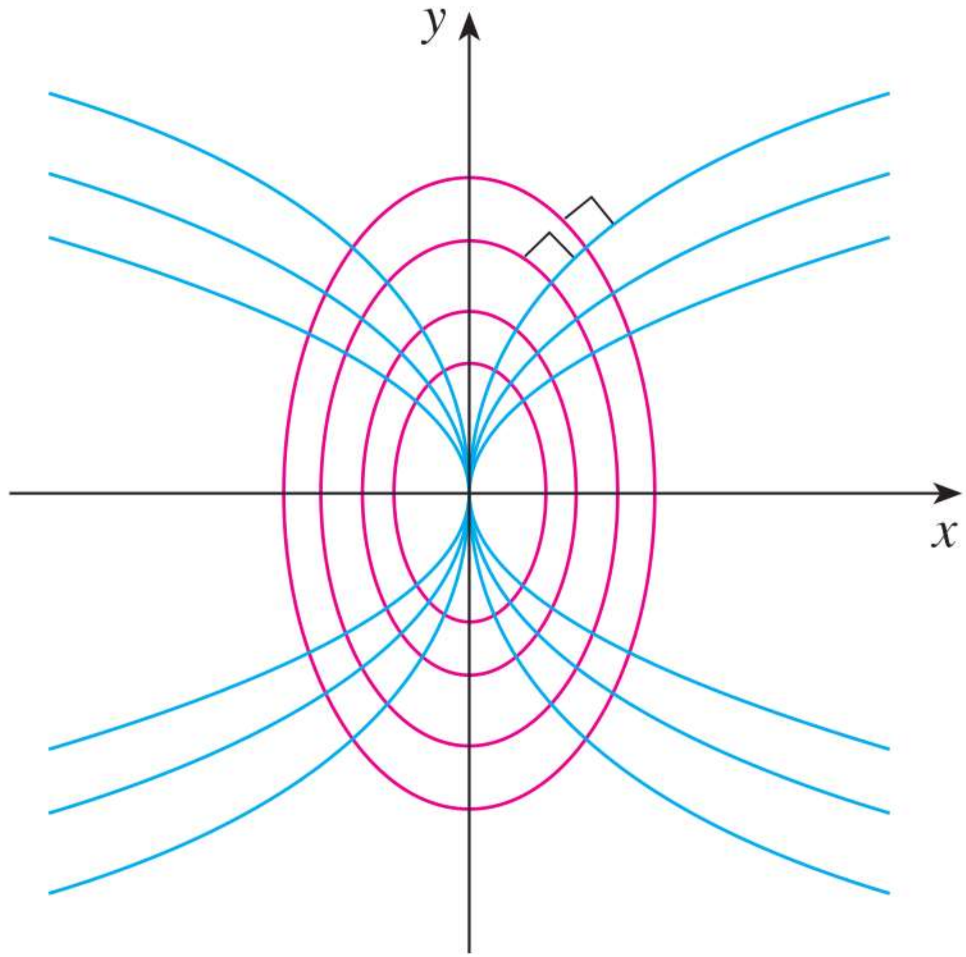
$$y\ dy = -2x\ dx$$

$$\frac{y^2}{2} = -x^2 + C$$

$$x^2 + \frac{y^2}{2} = C$$

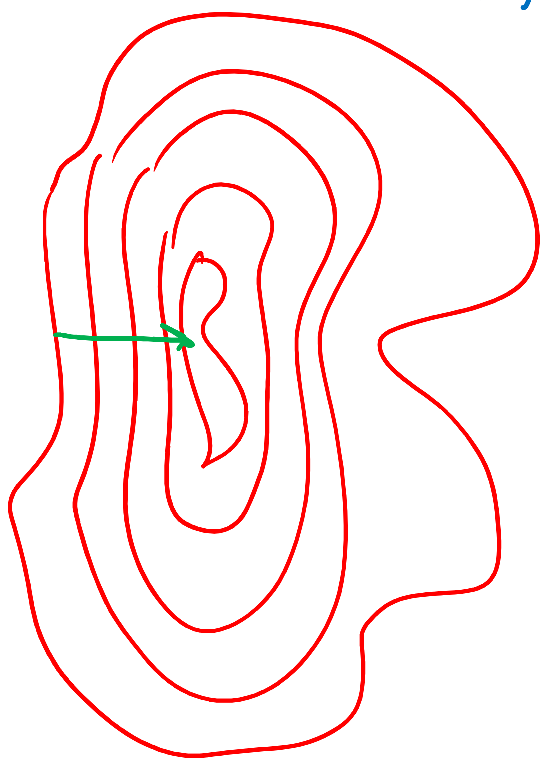
where C is a real number. Therefore the orthogonal trajectories of the family of curves $x = ky^2$ is $x^2 + \frac{y^2}{2} = C$. Note that the equation that we obtained describes ellipses centered at the origin.

$x = ky^2$ and $x^2 + \dfrac{y^2}{2} = C$ are orthogonal to each other.



7.3 Applications of Orthogonal Trajectories

Topographic map.



Each contour represents constant elevation

What's the shortest path to the top when starting on the outermost contour?

Observe that the shortest path is perpendicular to all of the contour lines. This is not a coincidence!