Daily Quiz

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7.1 Modeling with Differential Equations

- A differential equation is an equation that contains an unknown function and one or more of its derivatives.
- The overarching goal of studying differential equations is to find a specific function that satisfies the given differential equation.
- One model for growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population.
- This is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

7.1 Uninhibited Growth

- t = time (the independent variable)
- P = the number of individuals in the population (the dependent variable)
- The rate of growth of the population is the derivative dP/dt. So our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\frac{dP}{dt} = kP$$

where k is the proportionality constant.

1. Suppose $\frac{dP}{dt} = kP$; show that $P = e^{kt}$ is a solution to the given differential equation.

2. What is another solution to the above differential equation?

① Check if $P = e^{kt}$ satisfies $\frac{dP}{dt} = kP$:

$$\frac{d}{dt}(P) = \frac{d}{dt}(e^{kt})$$

$$\frac{dP}{dt} = ke^{kt}$$

$$\frac{dP}{dt} = kP \quad \checkmark$$

② $\frac{dP}{dt} = kP$ has many solutions:

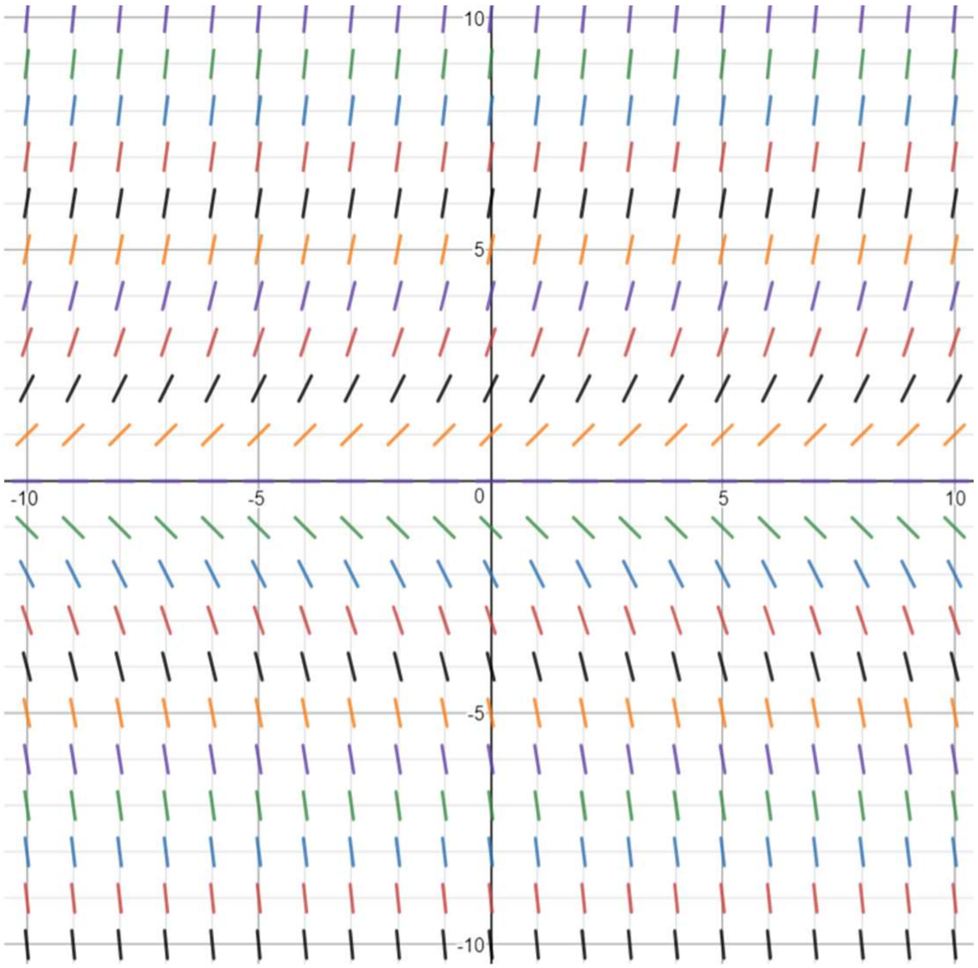
$P = 2e^{kt}$, $P = 100e^{kt}$, $P = 0$, $P = \pi e^{kt}$, $P = -4e^{kt}$

7.1 Uninhibited Growth

$$\frac{dP}{dt} = kP$$

- Can we find a function that satisfies the above differential equation?
- Yes. Take $P(t) = Ce^{kt}$ where C is a constant.
- For each value of C, we get a unique solution to the above differential equation.
- Allowing C to vary through all the real numbers, we get the family of solutions $P(t) = Ce^{kt}$

7.1 Uninhibited Growth



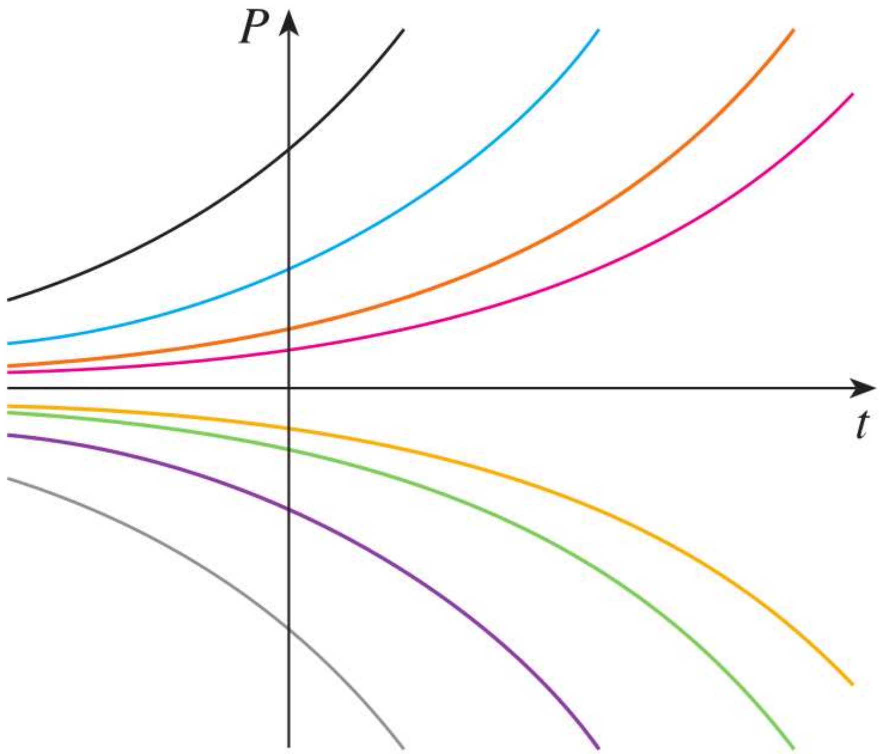


FIGURE 1

The family of solutions of $dP/dt = kP$

7.1 Logistic Growth (Inhibited Growth)

- How do we model for growth of a population with limited resources?
- Let's list the assumptions that we want our differential equation to reflect. Let M be the carrying capacity, which is the maximum sustainable population.

- $\frac{dP}{dt} \approx kP$ if P is small (Initially, the growth rate is proportional to P.)
- $\frac{dP}{dt} < 0$ if $P > M$ (P decreases if it ever exceeds M.)

A simple expression that incorporates both assumptions is given by the equation

2

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

7.1 Logistic Growth (Inhibited Growth)

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Notice that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. If $P > M$, then $1 - P/M$ is negative and so $dP/dt < 0$.

Equation 2 is called the *logistic differential equation*

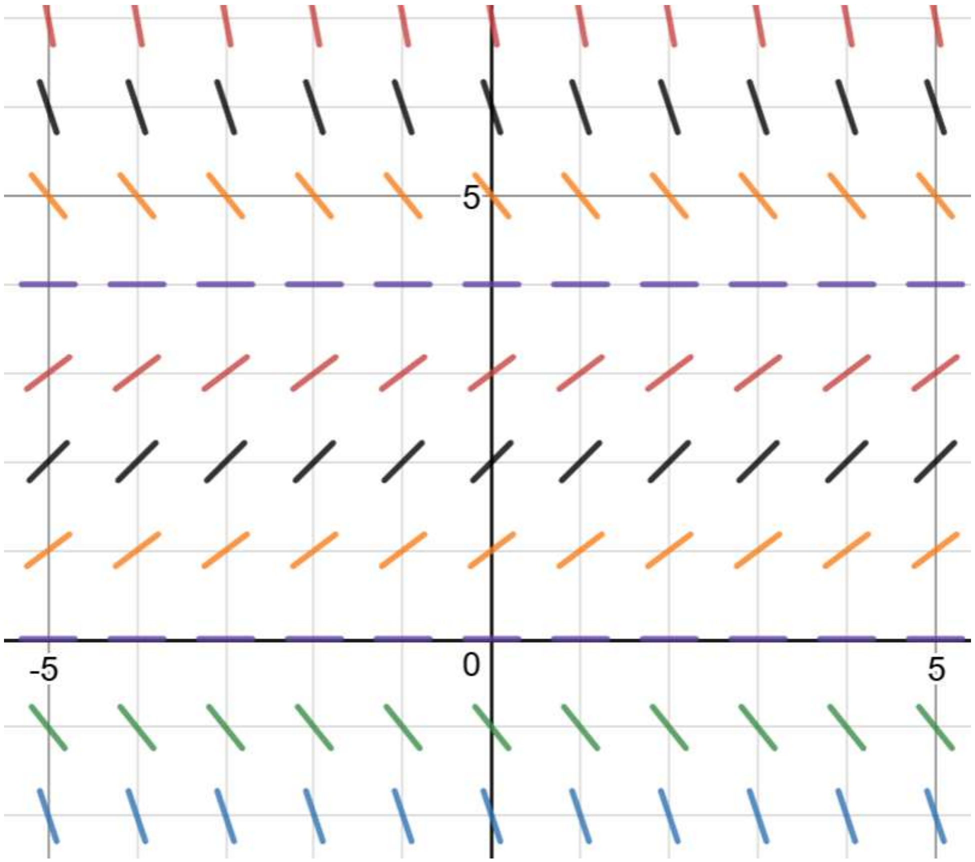
7.1 Logistic Growth (Inhibited Growth)

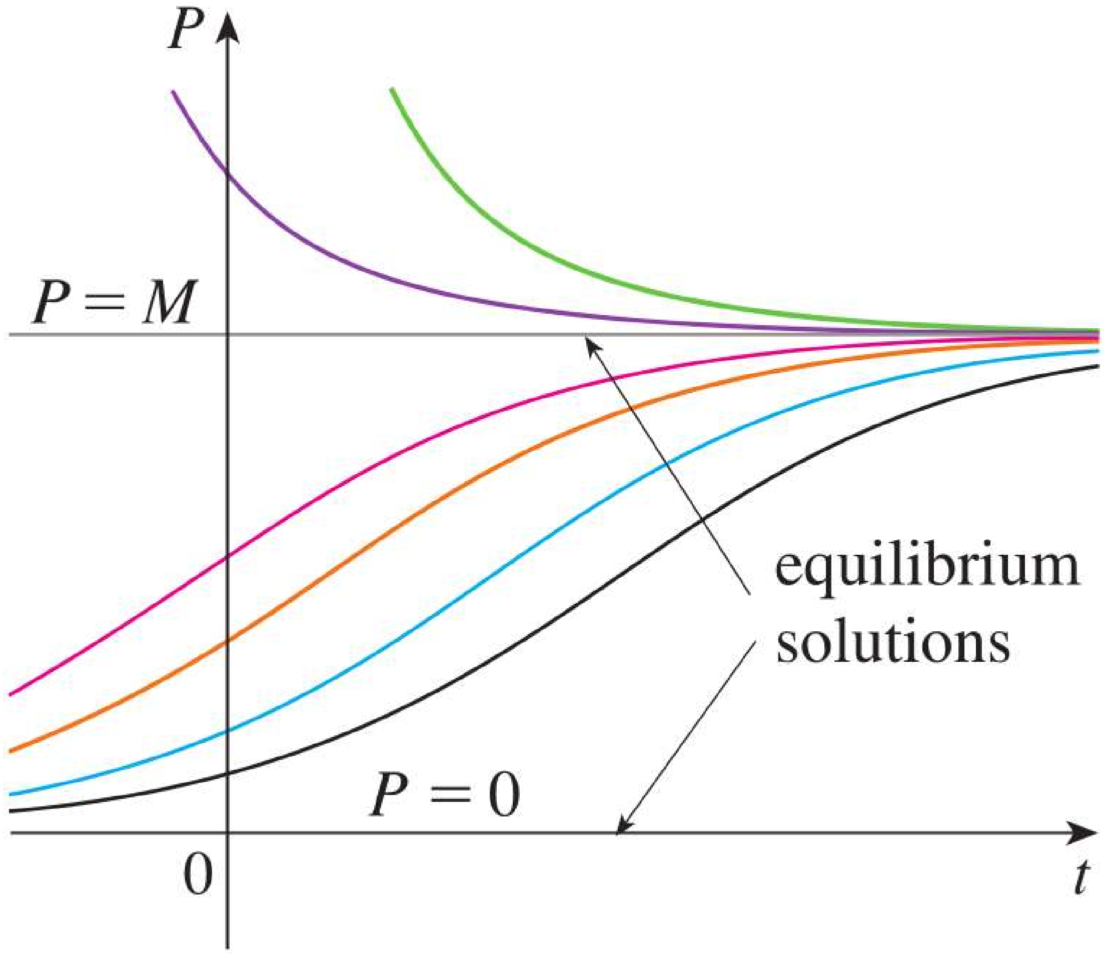
A simple expression that incorporates both assumptions is given by the equation

2
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

- Observe that P(t)=0 and P(t)=M are solutions of the above differential equation.
- The constant-valued solutions P(t) are called equilibrium solutions.

7.1 Logistic Growth (Inhibited Growth)





7.1 General Differential Equations

- The order of a differential equation is the order of the highest derivative that occurs in the equation.
- A function $y = f(x)$ is called a solution of a differential equation if the equation is satisfied when y and its derivatives y', y'', ... are substituted into the equation.
- When we solve a differential equation, we are expected to find all possible solutions of the equation, i.e. the family of solutions.
- Solving differential equations is a whole field in itself. Anything that changes over time can be modeled using differential equations.

V EXAMPLE 1 Verifying solutions of a differential equation Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

$$y' = \frac{d}{dt}\left(\frac{1+ce^t}{1-ce^t}\right)$$

$$= \frac{ce^t(1-ce^t)-(1+ce^t)(-ce^t)}{(1-ce^t)^2}$$

$$= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1-ce^t)^2}$$

$$= \frac{2ce^t}{(1-ce^t)^2}$$

Check the differential equation

$$y' = \frac{1}{2}(y^2-1)$$

$$\frac{2ce^t}{(1-ce^t)^2} = \frac{1}{2}\left[\left(\frac{1+ce^t}{1-ce^t}\right)^2 - 1\right]$$

$$= \frac{(1+ce^t)^2 - (1-ce^t)^2}{2(1-ce^t)^2}$$

$$= \frac{1+2ce^t+c^2e^{2t}-(1-2ce^t+c^2e^{2t})}{2(1-ce^t)^2}$$

$$= \frac{4ce^t}{2(1-ce^t)^2}$$

$$= \frac{2ce^t}{(1-ce^t)^2} \quad \checkmark$$

Therefore $y = \dfrac{1+ce^t}{1-ce^t}$ is a general solution to the differential equation

$$y' = \frac{1}{2}(y^2-1).$$

7.1 Initial-value Problems (IVP)

- In applications, finding general solutions (family of solutions) is usually not enough; we are often interested in finding a specific solution to a specific problem.
- For this, we are often given an initial condition of the form $y(t_0) = y_0$.
- The problem of finding a solution of the differential equation that satisfies the initial condition is called an initial-value problem.

Find a solution of the differential equation $y' = y$ that satisfies the initial condition $y(0) = 10$.

Recall: The differential equation $\frac{dy}{dx} = ky$ has as its solutions $y = Ce^{kx}$ where C is a constant. Let $k=1$ and since $y' = \frac{dy}{dx}$, the diff eq

$$y' = ky$$
$$y' = y$$

has as its solutions $y = Ce^{x}$. The initial condition $y(0) = 10$ means if we plug in $x=0$ to the function y, we get 10. The initial condition will give us the constant C:

$$y(0) = 10$$
$$Ce^{0} = 10$$
$$C = 10.$$

Therefore the solution to the above diff eq with the initial condition $y(0) = 10$ is $y = 10e^{x}$.

EXAMPLE 2 Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

From Example 1, $y = \frac{1+ce^t}{1-ce^t}$ is a general solution to the differential equation $y' = \frac{1}{2}(y^2-1)$.

we want to find a specific solution from this general solution by using the initial condition $y(0)=2$.

plug in 0:

$$2 = y(0) = \frac{1+Ce^0}{1-Ce^0} = \frac{1+C}{1-C}$$

$$\frac{1+c}{1-c} = 2$$

$$1+c = 2(1-c)$$

$$1+c = 2-2c$$

$$3c = 1$$

$$c = \frac{1}{3}$$

Hence

$$y = \frac{1+\frac{1}{3}e^t}{1-\frac{1}{3}e^t}$$

is the solution to the differential equation $y' = \frac{1}{2}(y^2-1)$ with the initial condition $y(0)=2$.

1. Consider the following differential equation:

$$y'' - y = 0.$$

For what values of r does the function $y = e^{rx}$ satisfy the differential equation?

2. If r_1 and r_2 are the values of r from part (a), show that $y = ae^{r_1 x} + be^{r_2 x}$ is a solution for any real number a and b.

① $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$y'' - y = 0$

$r^2 e^{rx} - e^{rx} = 0$

$(r^2 - 1) e^{rx} = 0$

Since e^{rx} is never 0 for finite values of r and x, (r^2-1) must be 0 in order for the equation to hold.

Hence $r^2 - 1 = 0$

$(r-1)(r+1) = 0$

$r = 1, -1.$

1. Consider the following differential equation:

$$y'' - y = 0.$$

For what values of r does the function $y = e^{rx}$ satisfy the differential equation?

2. If r_1 and r_2 are the values of r from part (a), show that $y = ae^{r_1 x} + be^{r_2 x}$ is a solution for any real number a and b.

(2) $r_1 = 1$, $r_2 = -1$

$y = ae^{r_1 x} + be^{r_2 x}$

$y = ae^{x} + be^{-x}$

Check if the y that we obtained satisfies the diff eq:

$y' = ae^{x} - be^{-x}$

$y'' = ae^{x} + be^{-x}$

$$y'' - y = 0$$

$$(ae^{x} + be^{-x}) - (ae^{x} + be^{-x}) = 0$$

$$0 = 0 \checkmark$$

Yes, $y = ae^{x} + be^{-x}$ is a solution for any real number a and b.