§5.10: Improper Integrals

Until now, we've considered integrals of continuous functions on closed intervals $[a,b]$. In this section, we will investigate a class of integrals, called improper integrals, where:

Type (1): one limit of integration is infinite

Type (2): the integrand is unbounded

Type (1):

<u>Example</u> 1: Find

$$\int_1^\infty \frac{1}{x^2}\,dx$$

To compute this improper integral we compute the definite integral,

$$\int_1^b \frac{1}{x^2}\,dx$$

where b is an arbitrary real number s.t. $b \geq 1$. That is,

$$\int_1^b \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^b = -\frac{1}{b} - \left(-\frac{1}{1}\right) = -\frac{1}{b} + 1$$

Then, we say that

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b \to \infty} \left(\int_1^{b} \frac{1}{x^2}\,dx \right)$$

$$= \lim_{b \to \infty} \left(-\frac{1}{b} + 1 \right)$$

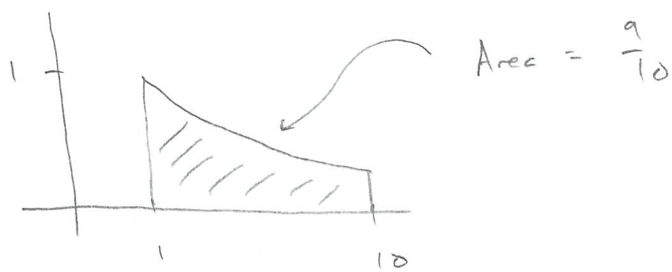
$$= 1$$

Since 1 is finite, we say that the improper integral converges to 1.

What's going on in pictures?

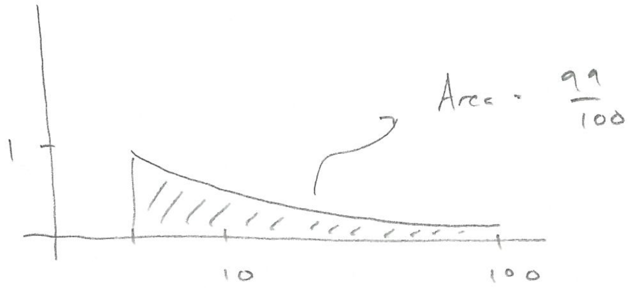
$b = 10$

$$\int_1^{10} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^{10} = -\frac{1}{10} - -\frac{1}{1} = \frac{9}{10}$$



$b = 100$

$$\int_1^{100} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^{100} = -\frac{1}{100} - -\frac{1}{1} = \frac{99}{100}$$



In summary, as $b \to \infty$, the area $\int_1^b \frac{1}{x^2}\,dx$ approaches 1.

An example of an improper integral which does not converge.

Example 2: $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Choose arbitrary $b \geq 1$. Then

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = \int_1^b x^{-1/2}\,dx = 2x^{1/2}\Big]_1^b = 2b^{1/2} - 2$$

As $b \to \infty$, $2b^{1/2} - 2 \to \infty$. So

$\int_1^b \frac{1}{\sqrt{x}}\,dx$ grows to infinity as $b \to \infty$

We say that $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ <u>diverges</u> b/c $\lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}}\,dx$

does not exist.

Remarks:

(i) Why is $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$ finite but $\lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}}\,dx$

not finite? B/c $\frac{1}{\sqrt{x}}$ does not decay fast enough

down to x-axis.

(ii) Note: $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$ but $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges.

L'hôspital's Rule w/ improper integral:

Example 3: Evaluate $\int_{-\infty}^0 x e^x\,dx$.

$$\int_{-\infty}^{0} x e^x\,dx = \lim_{a\to-\infty} \underbrace{\int_a^0 x e^x\,dx}_{(*)} \qquad (1)$$

Need to evaluate $(*)$:

$u = x \qquad v' = e^x$

$u' = 1 \qquad v = e^x$

$$(*) = -xe^x \Big]_a^0 + \int_0^a e^x \, dx$$

$$= -ae^a + \left(e^x \Big]_0^a \right)$$

$$= -ae^a + \left(e^a - e^0 \right)$$

$$= -ae^a + e^a - 1$$

So $$(1) = \lim_{a \to -\infty} \left(-ae^a + e^a + 1 \right)$$

Need to consider $\lim_{a \to -\infty} ae^a$:

$$\lim_{a \to -\infty} \frac{a}{\frac{1}{e^a}}$$

- $\lim_{a \to -\infty} a = -\infty$
- $\lim_{a \to -\infty} e^{-a} = -\infty$

$$= \lim_{a \to -\infty} \frac{a}{e^{-a}}$$

$$= \lim_{a \to -\infty} \frac{1}{-e^{-a}} = 0$$

So $$\lim_{a \to -\infty} \left(-ae^a + e^a + 1 \right) = 0 + 0 - 1 = -1$$

Example 4: Determine for which values of the exponent, p, the improper integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges.

Case 1: Suppose $p \neq 1$. Then

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left(\int_1^b x^{-p}\,dx\right) = \lim_{b\to\infty}\left(\frac{x^{-p+1}}{-p+1}\Big]_1^b\right)$$

$$= \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$

$$= \lim_{b\to\infty}\left(\left(\frac{1}{-p+1}\right)b^{-p+1} - \frac{1}{-p+1}\right)$$

This limit exists when $-p+1 \leq 0$. That is, when $1 \leq p$. Since $p \neq 1$, this limit exists when $1 < p$.

Note: The limit does not exist when $-p+1 > 0$. That is, when $p < 1$.

Case 2: Suppose $p = 1$. Then

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x}\,dx\right) = \lim_{b\to\infty}\left(\ln x\Big]_1^b\right)$$

$$= \lim_{b\to\infty} (\ln(b) - \ln(1)) = \lim_{b\to\infty} \ln(b) \to \infty$$ as

$b \to \infty$.

Hence, $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$, and diverges

for $p \leq 1$.

Q: What if both bounds are infinite?

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

Do two integrals separately -- must both converge for entire integral to converge!