§ 5.5 Integration by Substitution

Consider the function $h(x) = \sin(x^3)$. By the Chain Rule, recall that $h'(x) = \cos(x^3) \cdot 3x^2$.

Now, suppose one is asked to find

$$\int 3x^2 \cos(x^3)\, dx$$

Then since $h'(x) = 3x^2 \cos(x^3)$,

$$\int 3x^2 \cos(x^3)\, dx = h(x) + C$$

$$= \sin(x^3) + C$$

Notice that we have done the backwards chain rule.

The method of integration by substitution formalizes this backwards chain rule.

Example 1: Use substitution to find

$$\int 2t \cos(t^2)\, dt$$

Three steps:

(i) Choose a 'u' and find $\frac{du}{dt}$:

Let $u = t^2$

Then $\frac{du}{dt} = 2t$

Note: We set $u = t^2$ b/c its derivative, which is $2t$, also appears as a factor in the integrand!

(ii) Solve for dt:

$$\frac{du}{2t} = dt$$

(iii) Substitute:

$$\int 2t\cos(t^2)\,dt = \int \cancel{2t}\cos(u)\,\frac{du}{\cancel{2t}} = \int \cos(u)\,du$$

$$= \sin(u) + C = \sin(t^2) + C$$

Example 2: Use substitution to find

$$\int t e^{t^2+1}\,dt$$

(i) Let $u = t^2 + 1$

Then $\frac{du}{dt} = 2t$

Remark: Note that the derivative of u, which is $2t$, is not completely showing up in the integral. There is only a 't', so it is missing a constant factor 2. That's okay to have constant factors missing.

$$\frac{du}{2t} = dt$$

(iii) $$\int t e^{t^2+1}\,dt = \int \cancel{t} e^u \frac{du}{2\cancel{t}} = \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2+1} + C.$$

Example 3: Use substitution to find

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$$

(i) Let $u = \sqrt{x} = x^{1/2}$

Then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

(ii) $2\sqrt{x}\, du = dx$

(iii) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx = \int \frac{\sin(u)}{\cancel{\sqrt{x}}}\, 2\cancel{\sqrt{x}}\, du$

$= 2 \int \sin(u)\, du = -2\cos(u) + C$

$= -2\cos(\sqrt{x}) + C$

Definite integrals by substitution:

<u>Example</u> 4: Use substitution to compute

$$\int_{\theta=0}^{\theta=\pi/4} \frac{(\tan\theta)^3}{(\cos\theta)^2}\, d\theta$$

(i) Let $u = \tan\theta$

$\frac{du}{d\theta} = \frac{1}{(\cos\theta)^2}$

(ii) Solve for $d\theta$

$$(\cos\theta)^2 \, du = d\theta$$

(iii) Substitute (remember the bounds of integration!)

$$\int_{\theta=0}^{\theta=\pi/4} \frac{(\tan\theta)^3}{(\cos\theta)^2} \, d\theta = \int_{u=0}^{u=1} \frac{u^3}{(\cos\theta)^2} (\cos\theta)^2 \, du$$

When $\theta = 0$, $u = 0$

$\theta = \pi/4$ $u = \tan \pi/4 = 1$

$$= \int_{u=0}^{u=1} u^3 \, du = \frac{u^4}{4}\Big]_0^1 = \frac{1}{4}$$

Example 5: Suppose

$$\int_0^1 f(t)\,dt = 6$$

Compute $\displaystyle\int_{t=0}^{t=\frac{1}{10}} f(1-10t)\,dt$.

Let $u = 1 - 10t$. Then $\frac{du}{dt} = -10$

Hence, $\frac{du}{-10} = dt$

When $t = 0$, $u = 1 - 10(0) = 1$

$t = \frac{1}{10}$, $u = 1 - 10\left(\frac{1}{10}\right) = 0$

So $$\int_{t=0}^{t=1/10} f(1-10t)\,dt = \int_{u=1}^{u=0} f(u)\,\frac{du}{-10} = -\frac{1}{10}\int_1^0 f(u)\,du$$

$$= -\frac{1}{10}(-1)\int_0^1 f(u)\,du = \frac{1}{10}(6) = \frac{6}{10} = \frac{3}{5}$$