DEFORMATION THEORY

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1. Introduction and historical remarks

In mathematical deformation theory one studies how an object in a certain category of spaces can be varied in dependence of the points of a parameter space. In other words, deformation theory thus deals with the structure of families of objects like varieties, singularities, vector bundles, coherent sheaves, algebras or differentiable maps. Deformation problems appear in various areas of mathematics, in particular in algebra, algebraic and analytic geometry, and mathematical physics. According to Deligne, there is a common philosophy behind all deformation problems in characteristic zero. It is the goal of this survey to explain this point of view. Moreover, we will provide several examples with relevance for mathematical physics.

Historically, modern deformation theory has its roots in the work of Grothendieck, M. Artin, Quillen, Schlessinger, Kodaira–Spencer, Kuranishi, Deligne, Grauert, Gerstenhaber, and Arnol'd. The application of deformation methods to quantization theory goes back to Bayen–Flato–Fronsdal–Lichnerowicz–Sternheimer, and has lead to the concept of a star product on symplectic and Poisson manifolds. The existence of such star products has been proved by deWilde–Lecomte and Fedosov for symplectic and by Kontsevich for Poisson manifolds.

Recently, Fukaya and Kontsevich have found a far reaching connection between general deformation theory, the theory of moduli and mirror symmetry. Thus, deformation theory comes back to its origins, which lie in the desire to construct moduli spaces. Briefly, a moduli problem can be described as the attempt to collect all isomorphism classes of spaces of a certain type into one single object, the moduli space, and then to study its geometric and analytic properties. The observations by Fukaya and Kontsevich have lead to new insight into the algebraic geometry of mirror varieties and their application to string theory.

2. Basic definitions and examples

Deformation theory is based on the notion of a ringed space, so we briefly recall its definition.

Definition 2.1. Let $\mathbb{k}$ be a field. By a $\mathbb{k}$-*ringed space* one understands a topological space X together with a sheaf $\mathcal{A}$ of unital $\mathbb{k}$-algebras on X. The sheaf $\mathcal{A}$ will be called *structure sheaf* of the ringed space. In case each of the stalks $\mathcal{A}_x$, $x \in X$, is a local algebra, i.e. has a unique maximal ideal $\mathfrak{m}_x$, one calls $(X, \mathcal{A})$ a *locally* $\mathbb{k}$-*ringed space*. Likewise, one defines a *commutative* $\mathbb{k}$-*ringed space* as a ringed space such that the stalks of the structure sheaf are all commutative.

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Given two $\Bbbk$-ringed spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a *morphism* from $(X, \mathcal{A})$ to $(Y, \mathcal{B})$ is a pair (f, φ), where $f : X \to Y$ is a continuous mapping and $\varphi : f^{-1}\mathcal{B} \to \mathcal{A}$ a morphism of sheaves of algebras. This means in particular that for every point $x \in X$ there is a homomorphism of algebras $\varphi_x : \mathcal{B}_{f(x)} \to \mathcal{A}_x$ induced by φ. Under the assumption that both ringed spaces are local, (f, φ) is called a *morphism* of locally ringed spaces, if each φ_x is a homomorphism of local $\Bbbk$-algebras, i.e. maps the maximal ideal of $\mathcal{B}_{f(x)}$ to the one of $\mathcal{A}_x$.

Clearly, $\Bbbk$-ringed spaces (resp. locally or commutative $\Bbbk$-ringed spaces) together with their morphisms form a category. The following is a list of examples of ringed spaces, in particular of those which will be needed later.

Example 2.2.
(1) Denote by $\mathcal{C}^\infty$ the sheaf of smooth functions on $\mathbb{R}^n$, by $\mathcal{C}^\omega$ the sheaf of real analytic functions, and let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}^n$. Then $(\mathbb{R}^n, \mathcal{C}^\infty)$, $(\mathbb{R}^n, \mathcal{C}^\omega)$ and $(\mathbb{C}^n, \mathcal{O})$ are ringed spaces over $\mathbb{R}$ resp. $\mathbb{C}$.
(2) A *differentiable manifold* of dimension n can be understood as a locally $\mathbb{R}$-ringed space $(M, \mathcal{C}^\infty_M)$ which locally is isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty)$. Likewise, a *real analytic manifold* is a ringed space $(M, \mathcal{C}^\omega_M)$ which locally can be modelled by $(\mathbb{R}^n, \mathcal{C}^\omega)$, and a *complex manifold* is an $(M, \mathcal{O}_M)$ which locally looks like $(\mathbb{C}^n, \mathcal{O})$.
(3) Let D be a domain in $\mathbb{C}^n$, and $\mathcal{J}$ an ideal sheaf in $\mathcal{O}_D$ of finite type which means that $\mathcal{J}$ is locally finitely generated over $\mathcal{O}_D$. Let Y be the support of the quotient sheaf $\mathcal{O}_D/\mathcal{J}$. The pair $(Y, \mathcal{O}_Y)$, where $\mathcal{O}_Y$ denotes the restriction of $\mathcal{O}_D/\mathcal{J}$ to Y, then is a ringed space, called a *complex model space*. A *complex space* now is a ringed space $(X, \mathcal{O}_X)$ which locally looks like a complex model space (cf. Grauert–Remmert [6]).
(4) Let $\Bbbk$ be an algebraically closed field, and $\mathbb{A}^n$ the affine space over $\Bbbk$ of dimension n. Then $\mathbb{A}^n$ together with the sheaf of regular functions is a ringed space.
(5) Given a ring A, its spectrum $\operatorname{Spec} A$ together with the sheaf of regular functions $\mathcal{O}_A$ forms a ringed space (cf. [10, Sec. II.2]). One calls $(\operatorname{Spec} A, \mathcal{O}_A)$ an *affine scheme*. More generally, a *scheme* is a ringed space $(X, \mathcal{O}_X)$ which locally can be modelled by affine schemes.
(6) Finally, if A is a local $\Bbbk$-algebra, the pair $(*, A)$ can be understood as a locally ringed space. With A the algebra of formal power series $\Bbbk[[t]]$ over one variable t, this example plays an important role in the theory of formal deformations of algebras.

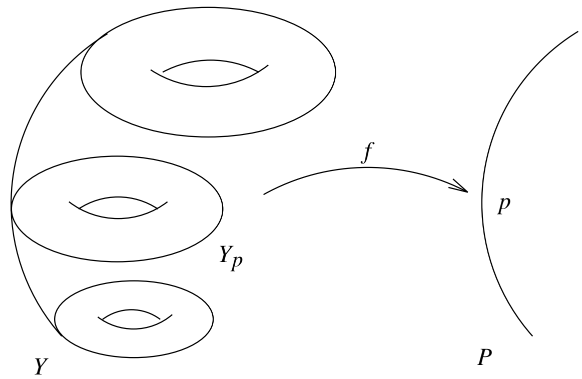
Definition 2.3. A morphism $(f, \varphi) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ of ringed spaces is called *fibered*, if the following conditions are fulfilled:
(1) $(P, \mathcal{S})$ is a commutative locally ringed space.
(2) $f : Y \to P$ is surjective.
(3) $\varphi_y : \mathcal{S}_{f(y)} \to \mathcal{B}_y$ maps $\mathcal{S}_{f(y)}$ into the center of $\mathcal{B}_y$ for each $y \in Y$.

The *fiber* of (f, φ) over a point $p \in P$ then is the ringed space $(Y_p, \mathcal{B}_p)$ defined by

$$Y_p = f^{-1}(p), \quad \mathcal{B}_p = \mathcal{B}_{|f^{-1}(p)}/\mathfrak{m}_p \, \mathcal{B}_{|f^{-1}(p)},$$

where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{S}_p$ which acts on $\mathcal{B}_{|f^{-1}(p)}$ via φ.

A fibered morphism of ringed spaces can be pictured in the following way:



Additionally to this intuitive picture, conditions (1) to (3) imply that the stalks $\mathcal{B}_y$ are central extensions of $\mathcal{B}_y/\mathfrak{m}_{f(y)}\mathcal{B}_y$ by $\mathcal{S}_{f(y)}$.

Definition 2.4. Let $(P, \mathcal{S})$ be a commutative locally ringed space over a field $\Bbbk$ with P connected, let $*$ be a fixed point in P, and $(X, \mathcal{A})$ a $\Bbbk$-ringed space. A *deformation* of $(X, \mathcal{A})$ over the *parameter space* $(P, \mathcal{S})$ with *distinguished point* $*$ then is a fibered morphism $(f, \varphi) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ over $\Bbbk$ together with an isomorphism $(i, \iota) : (X, \mathcal{A}) \to (Y_*, \mathcal{B}_*)$ such that for all $p \in P$ and $y \in f^{-1}(p)$ the homomorphism $\varphi_y : \mathcal{S}_p \to \mathcal{B}_y$ is flat.

The condition of flatness in the definition of a deformation serves as a substitute for "local triviality" and works also in the presence of singularities. See Palamodov [14, Sec. 3] for a discussion of this point.

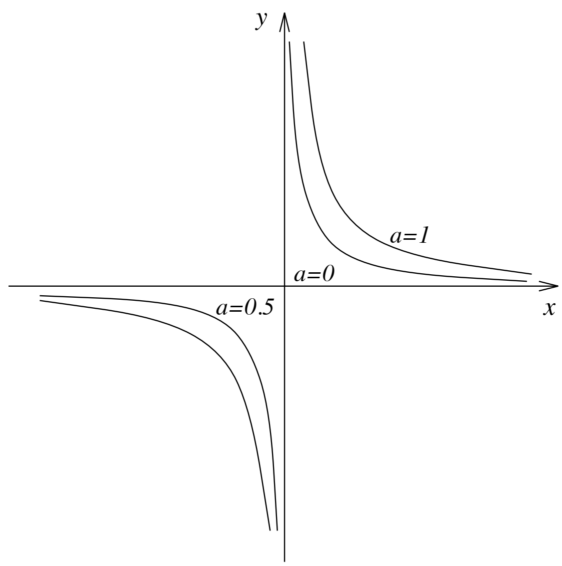
In the remainder of this section we now provide a list of some of the most important deformation problems in mathematics, and show how these can be formulated within the language from above.

2.1. **Products of $\Bbbk$-ringed spaces.** Let $(X, \mathcal{A})$ be any $\Bbbk$-ringed space and $(P, \mathcal{S})$ a $\Bbbk$-scheme. For any closed point $* \in P$ the product $(X \times P, \mathcal{B}) = (X, \mathcal{A}) \times_\Bbbk (P, \mathcal{S})$ then is a flat deformation of $(X, \mathcal{A})$ with distinguished point $*$. This can be seen easily from the fact that $\mathcal{B}_{(x,p)} = \mathcal{A}_x \otimes_\Bbbk \mathcal{S}_p$ for every $x \in X$ and $p \in P$.

2.2. **Families of matrices as deformations.** Let $(P, \mathcal{O}_P)$ be a complex space with distinguished point $*$ and $A_P : P \to \mathrm{Mat}(n \times n, \mathbb{C})$ a holomorphic family of complex $n \times n$-matrices over P. By the following construction, A_P can be understood as a deformation, more precisely as a deformation of the matrix $A := A_P(*)$. Let Y be the graph of A_P in the product space $P \times \mathrm{Mat}(n \times n, \mathbb{C})$ and $f : Y \to P$ be the restriction of the projection onto the first coordiante. Define the sheaf $\mathcal{B}$ as the inverse image sheaf $f^{-1}\mathcal{S}$, and let φ be the sheaf morphism which for every $y \in Y$ is induced be the identity map $\varphi_y : \mathcal{S}_{f(y)} \to \mathcal{B}_y := \mathcal{S}_{f(y)}$. It is then immediately clear that (f, φ) is a deformation of the fiber $f^{-1}(*)$ and that this fiber coincides with the matrix A.

Now let A be an arbitrary complex $n \times n$-matrix, and choose a $\mathrm{GL}(n, \mathbb{C})$-slice through A, i.e. a submanifold P containing A which is transversal to the $\mathrm{GL}(n, \mathbb{C})$-orbit through A. Hereby, it is assumed that $\mathrm{GL}(n, \mathbb{C})$ acts by the adjoint action on $\mathrm{Mat}(n \times n, \mathbb{C})$. The family A_P given by the canonical embedding $P \hookrightarrow \mathrm{Mat}(n \times n, \mathbb{C})$ now is a deformation of A. The germ of this deformation at $*$ is versal in the sense defined in the next section.

2.3. **Deformation of a scheme à la** GROTHENDIECK**.** Assume that $(P, \mathcal{S})$ is a connected scheme over $\mathbb{k}$. A deformation of a scheme $(X, \mathcal{A})$ then is a deformation $(f, \varphi) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ in the sense defined above together with the requirement that $f : Y \to P$ is a proper map, i.e. $f^{-1}(K)$ is compact for every compact $K \subset P$. As a particular example consider the $\mathbb{k}$-scheme $Y = \operatorname{Spec} \mathbb{k}[x, y, t]/(xy - t]$. It gives rise to a fibration $Y \to \operatorname{Spec} \mathbb{k}[t]$, whose fibers Y_a with $a \in \mathbb{k}$ are hyperbola $xy = a$, when $a \neq 0$, and consist of the two axes $x = 0$ and $y = 0$, when $a = 0$. For $\mathbb{k} = \mathbb{R}$ this deformation can be illustrated by the following picture.



For further information on this and similar examples see HARTSHORNE [10], in particular Example 3.3.2.

2.4. **Deformation of a complex space.** According to GROTHENDIECK one understands by a deformation of a complex space $(X, \mathcal{A})$ a morphism of complex spaces $(f, \varphi) : (Y, \mathcal{B}) \to (P, \mathcal{S})$ which is both a proper flat morphism of complex spaces and a deformation of $(X, \mathcal{A})$ as a ringed space. In case $(X, \mathcal{A})$ and $(P, \mathcal{S})$ are complex manifolds and if P is connected, each of the fibers Y_p is a compact complex manifold. Moreover, the family $(Y_p)_{p \in P}$ then is a family of compact complex manifolds in the sense of KODAIRA–SPENCER (cf. PALAMODOV [14]).

2.5. **Deformation of singularities.** Let p be a point of some $\mathbb{C}^n$. Two complex spaces $(X, \mathcal{O}_X) \subset (\mathbb{C}^n, \mathcal{O})$ and $(X', \mathcal{O}_{X'}) \subset (\mathbb{C}^n, \mathcal{O})$ with $x \in X \cap X'$ are then called *germ-equivalent* at x, if there exists an open neighborhood $U \in \mathbb{C}^n$ of x such that $X \cap U = X' \cap U$. Obviously, germ-equivalence at x is an equivalence relation indeed. We denote the equivalence class of X by $[X]_x$. Clearly, if $[X]_x = [X']_x$, then one has $\mathcal{O}_{X,x} = \mathcal{O}_{X',x}$ for the stalks at x. By a *singularity* one understands a pair $([X]_x, \mathcal{O}_{X,x})$. In the literature such a singularity is often denoted by (X, x). The singularity (X, x) is called *non-singular* or *regular*, if $\mathcal{O}_{X,x}$ is isomorphic to an algebra of convergent power series $\mathbb{C}\{z_1, \cdots, z_d\}$. A deformation of a complex singularity (X, x) over a complex germ $(P, *)$ is a morphism of ringed spaces $([Y]_x, \mathcal{O}_{Y,x}) \to ([P]_*, \mathcal{O}_{P,*})$ which is induced by a holomorphic map and which is a deformation of $([X]_x, \mathcal{O}_{X,x})$ as a ringed space. See Artin [1] and the overview article by Greuel [9] for further details and a variety of examples.

2.6. **First order deformation of algebras.** Consider a $\Bbbk$-algebra A and the truncated polynomial algebra $S = \Bbbk[\varepsilon]/\varepsilon^2\Bbbk[\varepsilon]$. Furthermore let $\alpha : A \times A \to A$ be a Hochschild 2-cocycle of A, in other words assume that the relation

$$a_1\, \alpha(a_2, a_3) - \alpha(a_1 a_2, a_3) + \alpha(a_1, a_2 a_3) - \alpha(a_1, a_2)\, a_3 \;=\; 0 \tag{2.1}$$

holds for all $a_1, a_2, a_3 \in A$. Then one can define a new k-algebra B, whose underlying linear structure is isomorphic to $A \otimes_{\Bbbk} S$ and whose product is given by the following construction: Any element $b \in B$ can be written uniquely in the form $b = a_0 + a_1\,\varepsilon$ with $a_0, a_1 \in A$. Then the product of $b = a_0 + a_1\,\varepsilon \in B$ and $b' = a'_0 + a'_1\,\varepsilon \in B$ is given by

$$b \cdot b' \;= a_0\, a'_0 + [\alpha(a_0, a'_0) + a_0 a'_1 + a_1 a'_0]\,\varepsilon. \tag{2.2}$$

By condition (2.1), this product is associative. One thus obtains a flat deformation $\Delta : S \to B$ of the algebra A and calls it the *first order* or *infinitesimal deformation* of A along the Hochschild cocycle α. For further information on this and the connection between deformation theory and Hochschild cohomology see the overview article [7] by Gerstenhaber–Schack.

2.7. **Formal deformation of an algebra.** Let us generalize the preceeding example and explain the concept of a formal deformation of an algebra by Gerstenhaber. Assume again A to be an arbitrary $\Bbbk$-algebra and choose bilinear maps $\alpha_n : A \times A \to A$ for $n \in \mathbb{N}$ such that α_0 is the product on A and α_1 is a Hochschild cocycle. Furthermore let S be the algebra $\Bbbk[[t]]$ of formal power series in one variable over $\Bbbk$. Then define on the linear space $B = A[[t]]$ of formal power series in one variable with coefficients in A the following bilinear map:

$$\star : B \times B \to B,\; \left(\sum_{n\in\mathbb{N}} a_n\, t^n, \sum_{n\in\mathbb{N}} b_n\, t^n \right) \mapsto \sum_{n\in\mathbb{N}} \sum_{\substack{k,l,m\in\mathbb{N}\\ k+l+m=n}} \alpha_m(a_k, b_l)\, t^n. \tag{2.3}$$

If B together with $\star$ becomes a $\Bbbk$-algebra or in other words if $\star$ is associative, one can easily see that it gives a flat deformation of A over $S = \Bbbk[[t]]$. In that case one says that B is a *formal deformation* of A by the family $(\alpha_n)_{n\in\mathbb{N}}$. Contrarily to the preceeding example there might not exist for every Hochschild cocycle α on A a formal deformation B of A defined by a family $(\alpha_n)_{n\in\mathbb{N}}$ such that $\alpha_1 = \alpha$. In case it exists, we will say the deformation B of A is *in direction* of α. If the third Hochschild cohomology group $\mathrm{H}^3(A, A)$ vanishes, there exists for every Hochschild cocycle α on A a deformation B of A in direction of α. See again Gerstenhaber–Schack [7] for further information.

2.8. **Formal deformation on symplectic and Poisson manifolds.** Let us consider the last two examples for the case, where A is the algebra $\mathcal{C}^\infty(M)$ of smooth functions on a symplectic or Poisson manifold M. Then the Poisson bracket $\{\ ,\ \}$ gives a Hochschild cocycle on $\mathcal{C}^\infty(M)$. There exists a first order deformation of $\mathcal{C}^\infty(M)$ along $\frac{1}{2i}\{\ ,\ \}$ and, even though $\mathrm{HH}^3(A, A)$ might not always vanish, a *deformation quantization* of M, that means a formal deformation of $\mathcal{C}^\infty(M)$ in direction of the Poisson bracket $\frac{1}{2i}\{\ ,\ \}$. For the symplectic case, this fact has been proven first by deWilde–Lecomte using methods from Hochschild cohomology theory. A more geometric and intuitive proof has been given by FEDOSOV. The Poisson case has been settled in the work of KONTSEVICH (see also Sec. 5.4).

2.9. **Quantized universal enveloping algebras according to** DRINFELD**.** A *quantized universal enveloping algebra* for a complex Lie algebra $\mathfrak{g}$ is a Hopf algebra A over $\mathbb{C}[[t]]$ such that A is a topologically free $\mathbb{C}[[t]]$-module (i.e. $A = (A/tA)[[t]]$ as left $\mathbb{C}[[t]]$-module) and A/tA is the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$. Because A is a topologically free $\mathbb{C}[[t]]$-module, A is a flat $\mathbb{C}[[t]]$-module and thus a deformation of $U\mathfrak{g}$ over $\mathbb{C}[[t]]$. See DRINFEL'D [4] and the monograph [11] by KASSEL for further details and examples of quantized universal enveloping algebras.

2.10. **Quantum plane.** Consider the tensor algebra $T = \bigoplus_{n\in\mathbb{N}} \left(\mathbb{R}^2\right)^{\otimes n}$ of the 2-dimensional real vector space $\mathbb{R}^2$, and let (x, y) be the canonical basis of $\mathbb{R}^2$. Then form the tensor product sheaf $\mathcal{T}_{\mathbb{C}^*} = T \otimes_{\mathbb{R}} \mathcal{O}_{\mathbb{C}^*}$ and let $\mathcal{I}_{\mathbb{C}^*}$ be the ideal sheaf in $\mathcal{T}_{\mathbb{C}^*}$ generated by the relation

$$x \otimes y - z\, y \otimes x \;=\; 0, \tag{2.4}$$

where $z : \mathbb{C}^* \to \mathbb{C}$ is the identity function. The quotient sheaf $\mathcal{B} = \mathcal{B}_{\mathbb{C}^*} = \mathcal{T}_{\mathbb{C}^*}/\mathcal{I}_{\mathbb{C}^*}$ then is a sheaf of $\mathbb{C}$-algebras and an $\mathcal{O}_{\mathbb{C}^*}$-module. Using Eq. (2.4) now move all occurances of x in an element of $\mathcal{B}_{\mathbb{C}^*}$ to the right of all y's. Since $\frac{1}{z}$ is an element of $\mathcal{O}(\mathbb{C}^*)$, one can thus show that $\mathcal{B}_{\mathbb{C}^*}$ is a free $\mathcal{O}_{\mathbb{C}^*}$-module. Hence $\mathcal{B}_{\mathbb{C}^*}$ is flat over $\mathcal{O}_{\mathbb{C}^*}$. Further it is easy to see that for every $q \in \mathbb{C}^*$ the $\mathbb{C}$-algebra $A_q = \mathcal{B}_q/\mathfrak{m}_q\mathcal{B}_q$ is freely generated by elements x, y with relations

$$x \otimes y - q\, y \otimes x \;=\; 0. \tag{2.5}$$

We call A_q the *q-deformed quantum plane* and $B = \mathcal{B}(\mathbb{C}^*)$ the *over $\mathbb{C}^*$ universally deformed quantum plane*. Altogether one can interpret B as a deformation of A_q over $\mathbb{C}^*$, in particular as a deformation of $A_1 = T \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x, y]$, the algebra of complex polynomials in two generators.

In the same way one can deform function algebras on higher dimensional vector spaces as well as function algebras on certain Lie groups. This way one obtains the quantum group $SU_q(2)$ as a deformation of a Hopf algebra of functions on $SU(2)$. See for example the work of FADDEEV–RESHETIKHIN–TAKHTAJAN, MANIN and WESS–ZUMINO for more information on q-deformations of vector spaces, Lie groups, differential calculi and all that.

3. Versal deformations

In this section and the following ones we consider only germs of deformations, i.e. deformations over parameter spaces of the form $(*, S)$. This means in particular that the structure sheaf only consists of its stalk S at $*$, a commutative local $\Bbbk$-algebra. Let us now suppose that the sheaf morphism $\varphi : (Y, \mathcal{B}) \to (*, S)$ (over the canonical map $Y \to *$) is a deformation of the ringed space $(X, \mathcal{A})$ and that $\tau : T \to S$ is a homomorphism of commutative local $\Bbbk$-algebras. Then the sheaf morphism $\tau^*\varphi : \mathcal{B} \otimes_S T \to T$ with $(\tau^*\varphi)_y(t) = 1 \otimes t$ for $y \in Y$ and $t \in T$ is a deformation of $(X, \mathcal{A})$ over the parameter space $(*, T)$. One says that the deformation $\tau^*\varphi$ is *induced* by the homomorphism τ.

Definition 3.1. A deformation $\varphi : (Y, \mathcal{B}) \to S$ of $(X, \mathcal{A})$ is called *versal*, if every (germ of a) deformation of $(X, \mathcal{A})$ is isomorphic to a deformation germ induced by a homomorphism of $\Bbbk$-algebras $\tau : T \to S$. A versal deformation is called *universal*, if the inducing homomorphism $\tau : T \to S$ is unique, and *miniversal*, if S is of minimal dimension.

Example 3.2.
(1) In Sec. 2.2, the construction of a versal deformation of a complex matrix A has been sketched.
(2) According to Kuranishi, every compact complex manifold has a versal deformation by an analytic germ. See [13] for a detailed exposition and Sec. 5.3 for a description of the principal ideas.
(3) Grauert has shown that for isolated singularities there exists a versal analytic deformation.
(4) By the work of Douady–Verdier, Grauert and Palamodov one knows that for every compact complex space there exists a miniversal analytic deformation. One of the essential methods in the existence proof hereby is Palamodov's construction of the cotangent complex (see [14]).
(5) Bingener [3] has further established Palomodov's approach and thus could provide a unified and quite general method for constructing versal deformations in analytic geometry.
(6) Fialowski–Fuchs have constructed miniversal deformations of Lie algebras.

4. Schlessinger's theorem

According to Grothendieck, spaces in Algebraic Geometry are represented by functors from a category of commutative rings to the category of sets. In this picture, an affine algebraic variety X over the base field $\Bbbk$ and with coordinate ring A is equivalently described by the functor $\mathrm{Hom}_{\mathrm{alg}}(A, -)$ defined on the category of commutative $\Bbbk$-algebras. As will be shown by examples in the next section, versal deformations are often encoded by functors representing spaces. More precisely, a deformation problem leads to a so-called *functor of Artin rings*, which means a covariant functor F from the category of (local) Artinian $\Bbbk$-algebras to the category of sets such that the set $F(\Bbbk)$ has exactly one element. The question now arises, under which conditions the functor F is *representable*, i.e. there exists a commutative $\Bbbk$-algebra A such that $F \cong \mathrm{Hom}_{\mathrm{alg}}(A, -)$. In the work of Schlessinger [15], the structure of functors of Artin rings has been studied in detail. Moreover, criteria have been established, when such a functor is *pro-prepresentable*, that means can be represented by a complete local algebra $\hat{A}$, where "completeness" is understood with respect to the $\mathfrak{m}$-adic topology. Beacuse of its importance for deformation theory, we will state Schlessinger's theorem in this section. Before we come to its details let us recall some notation.

Definition 4.1. By an Artinian $\Bbbk$-algebra over a field $\Bbbk$ one understands a commutative $\Bbbk$-algebra R which satisfies the following descending chain condition:

(Dec) Every descending chain $I_1 \supset \cdots \supset I_k \supset I_{k+1} \supset \cdots$ of ideals in R becomes stationary.

Among other, an Artinian algebra R has the following properties:

(1) R is noetherian, i.e. satisfies the ascending chain condition.
(2) Every prime ideal in R is maximal.
(3) (Chinese Remainder Theorem) R is isomorphic to a finite product $\Pi_{i=1}^{n} R_i$, where each R_i is a local Artinian algebra.

(4) Every maximal ideal $\mathfrak{m}$ of R is nilpotent, i.e. $\mathfrak{m}^k = 0$ for some $k \in \mathbb{N}$.
(5) Every quotient $R/\mathfrak{m}^k$ with $\mathfrak{m}$ maximal is finite dimensional.

Definition 4.2. Assume that $f : B \to A$ is a surjective homomorphism in the category $\mathbb{k}\text{-}\mathsf{Alg}_{\mathsf{l,Art}}$ of local Artinian $\mathbb{k}$-algebras. Then f is called a *small extension*, if $\ker f$ is a nonzero principal ideal (b) in B such that $\mathfrak{m}b = (0)$, where $\mathfrak{m}$ is the maximal ideal of B.

Theorem 4.3. (SCHLESSINGER [15, Thm. 2.11]) *Let F be a functor of Artin rings (over the base field $\mathbb{k}$). Assume that $A' \to A$ and $A'' \to A$ are morphisms in $\mathbb{k}\text{-}\mathsf{Alg}_{\mathsf{l,Art}}$, and consider the map*

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{4.1}$$

Then F is pro-representable, if and only if F has the following properties (H1) *to* (H4).

(H1) *The map* (4.1) *is a surjection, whenever $A'' \to A$ is a small extension.*
(H2) *The map* (4.1) *is a bijection, when $A = \mathbb{k}$ and $A'' = \mathbb{k}[\varepsilon]$.*
(H3) *One has $\dim_{\mathbb{k}}(t_F) < \infty$ for the tangent space $t_F := F(\mathbb{k}[\varepsilon])$.*
(H4) *For every small extension $A' \to A$, the map*

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$$

is an isomorphism.

Suppose that the functor F satisfies conditions (H1) to (H4), and let $\hat{A}$ be an arbitrary complete local $\mathbb{k}$-algebra. By Yoneda's lemma, every element $\xi = \operatorname{proj\,lim}_{n\in\mathbb{N}} \xi_n \in \hat{A} = \operatorname{proj\,lim}_{n\in\mathbb{N}} \hat{A}/\mathfrak{m}^n\hat{A}$ induces a natural transformation

$$\operatorname{Hom}_{\mathrm{alg}}(\hat{A}, -) \to F, \quad \big(u : \hat{A} \to R\big) \mapsto F(u_n)(\xi_n), \tag{4.2}$$

where $n \in \mathbb{N}$ is chosen large enough such that the homomorphism $u : \hat{A} \to R$ factors through some $u_n : \hat{A}/\mathfrak{m}^n \to R$. This is possible indeed, since R is Artinian. In the course of the proof of Schlessinger's theorem, $\hat{A}$ and the element $\xi \in \hat{A}$ are now constructed in such a way that (4.2) is an isomorphism.

5. Differential graded Lie algebras and deformation problems

According to a philosophy going back to DELIGNE "every deformation problem in characteristic 0 is controlled by a differential graded Lie algebra, with quasi-isomorphic differential graded Lie algebras giving the same deformation theory" (cf. GOLDMAN–MILLSON [8, p. 48]). In the following we will explain the main idea of this concept and apply it it to two particular examples.

5.1. Differential graded Lie algebras.

Definition 5.1. By a *graded algebra* over a field $\mathbb{k}$ one understands a graded $\mathbb{k}$-vector space $A^\bullet = \bigoplus_{k\in\mathbb{Z}} A^k$ together with a bilinear map

$$\mu : A^\bullet \times A^\bullet \to A^\bullet, \quad (a, b) \mapsto a \cdot b = \mu(a, b)$$

such that $A^k \cdot A^l \subset A^{k+l}$ for all $k, l \in \mathbb{Z}$. The graded algebra $A^\bullet$ is called *associative*, if $(ab)c = a(bc)$ for all $a, b, c \in A^\bullet$.

A *graded subalgebra* of $A^\bullet$ is a graded subspace $B^\bullet = \bigoplus_{k\in\mathbb{Z}} B^k \subset A^\bullet$ which is closed under μ, a *graded ideal* is a graded subalgebra $I^\bullet \subset A^\bullet$ such that $I^\bullet \cdot A^\bullet \subset I^\bullet$ and $A^\bullet \cdot I^\bullet \subset I^\bullet$.

A *homomorphism* between graded algebras $A^\bullet$ and $B^\bullet$ is a homogeneous map $f : A^\bullet \to B^\bullet$ of degree 0 such that $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in A^\bullet$.

From now on assume that $\Bbbk$ has characteristic $\neq 2, 3$. A *graded Lie algebra* then is a graded $\Bbbk$-vector space $\mathfrak{g}^\bullet = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^k$ together with a bilinear map

$$[\cdot,\cdot] : \mathfrak{g}^\bullet \times \mathfrak{g}^\bullet \to \mathfrak{g}^\bullet, \qquad (a, b) \mapsto [a, b]$$

such that the following axioms hold true:

(1) $[\mathfrak{g}^k, \mathfrak{g}^l] \subset \mathfrak{g}^{k+l}$ for all $k, l \in \mathbb{Z}$,
(2) $[\xi, \zeta] = -(-1)^{kl}\,[\zeta, \xi]$ for all $\xi \in \mathfrak{g}^k$, $\zeta \in \mathfrak{g}^l$.
(3) $(-1)^{k_1k_3}[[\xi_1, \xi_2], \xi_3] + (-1)^{k_2k_1}[[\xi_2, \xi_3], \xi_1] + (-1)^{k_3k_2}[[\xi_3, \xi_1], \xi_2] = 0$ for all $\xi_i \in \mathfrak{g}^{k_i}$ with $i = 1, 2, 3$.

By axiom (1) it is clear that a graded Lie algebra is in particular a graded algebra. So the above defined notions of a graded ideal, homomorphism, etc. apply as well to graded Lie algebras.

Example 5.2. Let $A^\bullet = \bigoplus_{k\in\mathbb{Z}} A^k$ be a graded associative algebra. Then $A^\bullet$ becomes a graded Lie algebra with the bracket

$$[a, b] = ab - (-1)^{kl}ba \quad \text{for } a \in A^k \text{ and } b \in A^l.$$

The space $A^\bullet$ regarded as a graded Lie algebra is often denoted by $\mathfrak{lie}^\bullet(A^\bullet)$.

Definition 5.3. A linear map $D : A^\bullet \to A^\bullet$ defined on a graded algebra $A^\bullet$ is called a *derivation* of *degree* l, if

$$D(ab) = (Da)b + (-1)^{kl}a(Db) \quad \text{for all } a \in A^k \text{ and } b \in A^\bullet.$$

A graded (Lie) algebra $A^\bullet$ together with a derivation d of degree 1 is called a *differential graded (Lie) algebra* if $d \circ d = 0$. Then $(A^\bullet, d)$ becomes a cochain complex. Since $\ker d$ is a graded subalgebra of $A^\bullet$ and $\operatorname{im} d$ a graded ideal in $\ker d$, the cohomology space

$$H^\bullet(A^\bullet, d) = \ker d/\operatorname{im} d$$

inherits the structure of a graded (Lie) algebra from $A^\bullet$.

Let $f : A^\bullet \to B^\bullet$ be a homomorphism of differential graded (Lie) algebras $(A^\bullet, d)$ and $(B^\bullet, \partial)$. Assume further that f is a cochain map, i.e. that $f \circ d = \partial \circ f$. Then one calls f a *quasi-isomorphism* or says that the differential graded (Lie) algebras $A^\bullet$ and $B^\bullet$ are *quasi-isomorphic*, if the induced homomorphism on the cohomology level $\overline{f} : H^\bullet(A^\bullet, d) \to H^\bullet(B^\bullet, \partial)$ is an isomorphism. Finally, a differential graded (Lie) algebra $(A^\bullet, d)$ is called *formal*, if it is quasi-isomorphic to its cohomology $(H^\bullet(A^\bullet, d), 0)$.

5.2. **Maurer–Cartan equation.** Assume that $(\mathfrak{g}^\bullet, [\cdot,\cdot], d)$ is a differential graded Lie algebra over $\mathbb{C}$. Define the space $\mathcal{MC}(\mathfrak{g}^\bullet)$ of solutions of the Maurer–Cartan equation by

$$\mathcal{MC}(\mathfrak{g}^\bullet) := \big\{\omega \in \mathfrak{g}^1 \mid d\omega - \frac{1}{2}[\omega, \omega] = 0\big\}. \tag{5.1}$$

In case the differential graded Lie algebra $\mathfrak{g}^\bullet$ is nilpotent, this space naturally possesses a groupoid structure, or in other words a set of arrows which are all invertible. The reason for this is that under the assumption of nilpotency, the space $\mathfrak{g}^0$ is equipped with the Campbell–Hausdorff multiplication

$$\mathfrak{g}^0 \times \mathfrak{g}^0 \to \mathfrak{g}^0, \quad (X, Y) \mapsto \log(\exp X, \exp Y),$$

and the group $\mathfrak{g}^0$ acts on $\mathfrak{g}^1$ by the exponential function. More precisely, in this situation one can define for two objects $\alpha, \beta \in \mathcal{MC}(\mathfrak{g}^\bullet)$ the space of arrows $\alpha \to \beta$ as the set of all $\lambda \in \mathfrak{g}^0$ such that $\exp \lambda \cdot \alpha = \beta$.

We have now the means to define for every complex differential graded Lie algebra $\mathfrak{g}^\bullet$ its *deformation functor* $\mathrm{Def}_{\mathfrak{g}^\bullet}$. This functor maps the category of local Artinian $\mathbb{C}$-algebras to the category of groupoids and is defined on objects as follows:

$$\mathrm{Def}_{\mathfrak{g}^\bullet}(R) := \mathcal{MC}(\mathfrak{g}^\bullet \otimes \mathfrak{m}). \tag{5.2}$$

Hereby, R is a complex local Artinian algebra, and $\mathfrak{m}$ its maximal ideal. Note that since R is Artinian, $\mathfrak{g}^\bullet \otimes \mathfrak{m}$ is a nilpotent differential graded Lie algebra, hence $\mathrm{Def}_{\mathfrak{g}^\bullet}(R)$ carries a groupoid structure as constructed above. Clearly, $\mathrm{Def}_{\mathfrak{g}^\bullet}$ is also a functor of Artin rings as defined in the previous section.

With appropriate choices of the differential graded Lie algebra $\mathfrak{g}^\bullet$, essentially all deformation problems from Section 2 can be recovered via a functor of the form $\mathrm{Def}_{\mathfrak{g}^\bullet}$. Below, we will show in some detail how this works for two examples, namely the deformation theory of complex manifolds and the deformation quantization of Poisson manifolds. But before we come to this, let us state a result which shows how the deformation functor behaves under quasi-isomorphisms of the underlying differential graded Lie algebra. This result is crucial in a sense that it allows to equivalently describe a deformation problem with controlling $\mathfrak{g}^\bullet$ by any other differential graded Lie algebra within the quasi-isomorphism class of $\mathfrak{g}^\bullet$. So in particular in the case, where the differential graded Lie algebra is formal, one often obtains a direct solution of the deformation problem.

Theorem 5.4. (Deligne, Goldman–Millson) *Assume that $f : \mathfrak{g}^\bullet \to \mathfrak{h}^\bullet$ is a quasi-isomorphism of differential graded Lie algebras. For every local Artinian $\mathbb{C}$-algebra R the induced functor $f_* : \mathrm{Def}_{\mathfrak{g}^\bullet}(R) \to \mathrm{Def}_{\mathfrak{h}^\bullet}(R)$ then is an equivalence of groupoids.*

5.3. **The Kodaira–Spencer algebra controlling deformations of compact complex manifolds.** Let M be a compact complex n-dimensional manifold. Recall that then the complexified tangent bundle $T_{\mathbb{C}}M$ has a decomposition into a holomorphic tangent bundle $T^{1,0}M$ and an antiholomorphic tangent bundle $T^{0,1}M$. This leads to a decomposition of the space of complex n-forms into the spaces $\Omega^{p,q}M$ of forms on M of type (p,q). More generally, a smooth subbundle $J^{0,1} \subset T_{\mathbb{C}}M$ which induces a decomposition of the form $T_{\mathbb{C}}M = J^{1,0} \oplus J^{0,1}$, where $J^{1,0} := \overline{J^{0,1}}$, is called an *almost complex structure* on M. Clearly, the decomposition of $T_{\mathbb{C}}M$ into the holomorphic and antiholomorphic part is an almost complex structure, and an almost complex structure which is induced by a complex structure is called *integrable*. Assume that an almost complex structure $J^{0,1}$ is given on M and that it has *finite distance* to the complex structure on M. The latter means that the restriction $\varrho_J^{0,1}$ of the projection $\varrho : T_{\mathbb{C}}M \to T^{0,1}M$ along $T^{1,0}M$ to the subbundle $J^{0,1}$ is an isomorphism. Denote by β the inverse of $\varrho_J^{0,1}$, and let $\omega \in \Omega^{0,1}(M, T^{1,0}M)$ be the composition $-\varrho \circ \beta$. One checks immediately that every almost complex structure with finite distance to the complex structure on M is uniquely characterized by a section $\omega \in \Omega^{0,1}(M, T^{1,0}M)$ and that every element of $\Omega^{0,1}(M, T^{1,0}M)$ comes from an almost complex structure on M.

As a consequence of the Newlander–Nirenberg theorem one can now show that the almost complex structure $J^{0,1}$ resp. ω is integrable, if and only if the equation

$$\overline{\partial}\omega - \frac{1}{2}[\omega,\omega] = 0 \tag{5.3}$$

is fulfilled. But this is nothing else than the Maurer–Cartan equation in the Kodaira–Spencer differential graded Lie algebra

$$\big(\mathfrak{L}^\bullet, \overline{\partial}, [\cdot,\cdot]\big) = \Big(\bigoplus_{p\in\mathbb{N}} \Omega^{0,p}(M, T^{1,0}M), \overline{\partial}, [\cdot,\cdot]\Big).$$

Hereby, $\Omega^{0,p}(M,T^{1,0}M)$ denotes the $T^{1,0}M$-valued differential forms on M of type $(0,p)$, $\overline{\partial} : \Omega^{0,p}(M,T^{1,0}M) \to \Omega^{0,p+1}(M,T^{1,0}M)$ the Dolbeault operator, and $[\cdot,\cdot]$ is induced by the Lie bracket of holomorphic vector fields. As a consequence of these considerations, deformations of the complex manifold M can equivalently be described by families $(\omega_p)_{p\in P} \subset \mathfrak{L}^1$ which satisfy Eq. (5.3) and $\omega_* = 0$. Thus it remains to determine the associated deformation functor $\mathrm{Def}_{\mathfrak{L}^\bullet}$.

According to Schlessinger's theorem, the functor $\mathrm{Def}_{\mathfrak{L}^\bullet}$ is pro-representable. Hence there exists a local $\mathbb{C}$-algebra $R_{\mathfrak{L}^\bullet}$ complete with respect to the $\mathfrak{m}$-adic topology such that

$$\mathrm{Def}_{\mathfrak{L}^\bullet}(R) = \mathrm{Hom}_{\mathrm{alg}}(R_{\mathfrak{L}^\bullet}, R) \tag{5.4}$$

for every local Artinian $\mathbb{C}$-algebra R. Moreover, by the theorem of M. Artin, there exists a "convergent" solution of the Maurer–Cartan equation, i.e. $R_{\mathfrak{L}^\bullet}$ can be replaced in Eq. (5.4) by a ring $\overline{R}_{\mathfrak{L}^\bullet}$ representing an analytic germ.

Theorem 5.5. (Kodaira–Spencer, Kuranishi) *The ringed space $\big(\overline{R}_{\mathfrak{L}^\bullet}, (0)\big)$ is a miniversal deformation of the complex structure on M.*

5.4. **Deformation quantization of Poisson manifolds.** Let A be an associative $\mathbb{k}$-algebra with $\operatorname{char}\mathbb{k} = 0$. Put for every integer $k \geq -1$

$$\mathfrak{g}^k := \mathrm{Hom}_{\mathbb{k}}(A^{\otimes(k+1)}, A).$$

Then $\mathfrak{g}^\bullet$ becomes a graded vector space. Let us impose a differential and a bracket on $\mathfrak{g}^\bullet$. The differential is the usual *Hochschild coboundary* $b : \mathfrak{g}^k \to \mathfrak{g}^{k+1}$,

$$\begin{aligned} bf(a_0 \otimes \ldots \otimes a_{k+1}) &:= a_0\, f(a_1 \otimes \ldots \otimes a_{k+1}) + \\ &+ \sum_{i=0}^{k} (-1)^{i+1} f(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{k+1}) + (-1)^k f(a_0 \otimes \ldots \otimes a_k)\, a_{k+1}. \end{aligned}$$

The bracket is the *Gerstenhaber bracket*

$$[\cdot,\cdot] : \mathfrak{g}^{k_1} \times \mathfrak{g}^{k_2} \to \mathfrak{g}^{k_1+k_2}, \quad [f_1, f_2] := f_1 \circ f_2 - (-1)^{k_1 k_2} f_2 \circ f_1,$$

where $f_1 \circ f_2\,(a_0 \otimes \ldots \otimes a_{k1+k_2}) :=$

$$:= \sum_{i=0}^{k_1} (-1)^{ik_2} f_1\big(a_0 \otimes \ldots \otimes a_{i-1} \otimes f_2(a_i \otimes \ldots \otimes a_{i+k_2}) \otimes a_{i+k_2+1} \otimes \ldots \otimes a_{k_1+k_2}\big).$$

The triple $(\mathfrak{g}^\bullet, b, [\cdot,\cdot])$ then is a differential graded Lie algebra.

Consider the Maurer–Cartan equation $b\gamma - \frac{1}{2}[\gamma,\gamma] = 0$ in $\mathfrak{g}^1$. Obviously, it is equivalent to the equality

$$\begin{aligned} &a_0\gamma(a_1,a_2) - \gamma(a_0a_1,a_2) + \gamma(a_0,a_1a_2) - \gamma(a_0,a_1)a_2 = \\ &\qquad = \gamma(\gamma(a_0,a_1),a_2) - \gamma(a_0,\gamma(a_1,a_2)) \quad \text{for } a_0,a_1,a_2 \in A. \end{aligned} \tag{5.5}$$

If one defines now for some $\gamma \in \mathfrak{g}^1$ the bilinear map $m : A \times A \to A$ by $m(a,b) = ab + \gamma(a,b)$, then (5.5) implies that m is associative, if and only if γ satisfies the Maurer–Cartan equation.

Let us apply these observations to the case, where A is the algebra $\mathcal{C}^\infty(M)[[t]]$ of formal power series in one variable with coefficients in the space of smooth functions on a Poisson manifold M. By (a variant of) the theorem of Hochschild–Kostant–Rosenberg and Connes one knows that in this case the cohomology of $(\mathfrak{g}^\bullet, b)$ is given by formal power series with coefficients in the space $\Gamma^\infty(\Lambda^\bullet TM)$ of antisymmetric vector fields. Now, $\Gamma^\infty(\Lambda^\bullet TM)$ carries a natural Lie algebra bracket as well, namely the Schouten bracket. Thus, one obtains a second differential graded Lie algebra $(\Gamma^\infty(\Lambda^\bullet TM)[[t]], 0, [\cdot,\cdot])$. Unfortunately, the projection onto cohomology $(\mathfrak{g}^\bullet, b) \to \Gamma^\infty(\Lambda^\bullet TM)[[t]]$ does not preserve the natural brackets, hence is not a quasi-isomorphism in the category of differential graded Lie algebras. It has been the fundamental observation by Kontsevich that this defect can be cured as follows.

Theorem 5.6. (Kontsevich [12]) *For every Poisson manifold M the differential graded Lie algebra $(\mathfrak{g}^\bullet, b, [\cdot,\cdot])$ is formal in the sense that there exists a quasi-isomorphism $(\mathfrak{g}^\bullet, b, [\cdot,\cdot]) \to (\Gamma^\infty(\Lambda^\bullet TM)[[t]], 0, [\cdot,\cdot])$ in the category of L^∞-algebras.*

Note that the theorem only claims the existence of a quasi-isomorphism in the category of L^∞-algebras or in other words in the category of homotopy Lie algebras. This is a notion somewhat weaker than a differential graded Lie algebra, but Thm. 5.4 also holds in the context of L^∞-algebras.

Since the solutions of the Maurer-Cartan equation in $(\Gamma^\infty(\Lambda^\bullet TM)[[t]], 0, [\cdot,\cdot])$ are exactly the formal paths of Poisson bivector fields on M, Kontsevich's formality theorem entails

Corollary 5.1. *Every Poisson manifold has a formal deformation quantization.*

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