# MATH 4450/5450

SEBASTIAN CASALAINA

## 1. Exercises

1.1. **Exercise (3) (a). Prove the following theorem** using the outline below.

**Theorem 1.1.** *Let $f : U \to \mathbb{R}$ be a continuous function on an open set $U \subset \mathbb{R}^2$. Fixing coordinates $x$ and $y$ on $\mathbb{R}^2$, suppose further that $\partial f/\partial x : U \to \mathbb{R}$ exists and is also continuous. Then given a closed set $V = [x_0, x_1] \times [y_0, y_1] \subseteq U$, for any $x \in [x_0, x_1]$ we have*

$$\frac{d}{dx}\int_{y_0}^{y_1} f(x,y)dy = \int_{y_0}^{y_1} \frac{\partial f(x,y)}{\partial x} dy.$$

[Hint: use the fact from multivariable calculus that if

$$f : [x_0, x_1] \times [y_0, y_1] \to \mathbb{R}$$

is continuous, then

$$\int_{y_0}^{y_1} \left( \int_{x_0}^{x_1} f(x,y)dx \right) dy = \int_{x_0}^{x_1} \left( \int_{y_0}^{y_1} f(x,y)dy \right) dx,$$

and rewrite the latter integral in the theorem as

$$\int_{y_0}^{y_1} \frac{\partial f(x,y)}{\partial x} dy = \frac{d}{dx}\int_{x_0}^{x} \left( \int_{y_0}^{y_1} \frac{\partial f(t,y)}{\partial t} dy \right) dt.]$$

1.2. **Exercise (3) (b).** WARNING: It is tempting to write

$$\begin{aligned} \frac{d}{dx}\int_{y_0}^{y_1} f(x,y)dy &= \lim_{h\to 0} \frac{1}{h} \left( \int_{y_0}^{y_1} f(x+h,y) - \int_{y_0}^{y_1} f(x,y)dy \right) \\ &= \lim_{h\to 0} \left( \int_{y_0}^{y_1} \frac{\partial f(t,y)}{\partial t} dy + \int_{y_0}^{y_1} r(x,y,h)dz \right) \\ &= \int_{y_0}^{y_1} \frac{\partial f(t,y)}{\partial t} dy + \lim_{h\to 0} \int_{y_0}^{y_1} r(x,y,h)dz \end{aligned}$$

where

$$r(x,y,h) = \frac{f(x+h,y) - f(x,y)}{h} - \frac{\partial f(t,y)}{\partial t}$$

and

$$\lim_{h\to 0} r(x,y,h) = 0.$$

But recall, and this is the warning, *it is not always the case that if a function converges to zero, that its integral converges to zero.* [One needs a stronger assumption such as uniform convergence.]

As an exercise, **prove the following proposition**:

**Proposition 1.2.** *There exists a continuous function* $f : (0,1]\times\mathbb{R} \to \mathbb{R}$ *such that for each fixed* $x \in \mathbb{R}$, $\lim_{h\to 0^+} f(h,x) = 0$, *but*

$$\lim_{h\to 0^+} \int_0^2 f(h,x)dx \neq 0.$$

[Hint: Consider the function

$$g(h,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ hx & \text{if } 0 \leq x \leq 1/h \\ 2-hx & \text{if } 1/h \leq x \leq 2/h \\ 0 & \text{if } 2/h \geq x \end{cases}$$

and set $f(h,x) = hg(h,x)$.]