CHAPTER 6

# Vector spaces and linear maps

In what follows, fix $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. More generally, $K$ can be any field.

## 1. Vector spaces

**Definition 6.1.** *A **vector space** over K consists of a triple $(V, +, \cdot)$, where V is a set, and $+ : V \times V \to V$ and $\cdot : K \times V \to V$ are maps, satisfying the following properties:*

(1) *(Group laws)*
  (a) *(Additive identity) There exists an element $O \in V$ such that for all $v \in V$, $v + O = v$;*
  (b) *(Additive inverse) For each $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = O$;*
  (c) *(Associativity of addition) For all $v_1, v_2, v_3 \in V$,*
$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3);$$
(2) *(Abelian property)*
  (a) *(Commutativity of addition) For all $v_1, v_2 \in V$,*
$$v_1 + v_2 = v_2 + v_1;$$
(3) *(Module conditions)*
  (a) *For all $\lambda \in K$ and all $v_1, v_2 \in V$,*
$$\lambda \cdot (v_1 + v_2) = (\lambda \cdot v_1) + (\lambda \cdot v_2).$$
  (b) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 + \lambda_2) \cdot v = (\lambda_1 \cdot v) + (\lambda_2 \cdot v).$$
  (c) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 \lambda_2) \cdot v = \lambda_1 \cdot (\lambda_2 \cdot v).$$
  (d) *For all $v \in V$,*
$$1 \cdot v = v.$$

*In the above, for all $\lambda \in K$ and all $v, v_1, v_2 \in V$ we have denoted $+(v_1, v_2)$ by $v_1 + v_2$ and $\cdot(\lambda, v)$ by $\lambda \cdot v$.*

In addition, for brevity, we will often write $\lambda v$ for $\lambda \cdot v$.

EXAMPLE 6.2 (The vector space $K^n$). By definition,
$$K^n = \{(x_1, \dots, x_n) \,:\, x_i \in K,\ 1 \le i \le n\}.$$
The map $+ : K^n \times K^n \to K^n$ is defined by the rule
$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in K^n$. The map $\cdot : K \times K^n \to K^n$ is defined by the rule

$$\lambda \cdot (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

for all $\lambda \in K$ and $(x_1, \dots, x_n) \in K^n$.

**Exercise 6.3.** *Show that $(K^n, +, \cdot)$, defined in the example above, is a vector space.*

**Exercise 6.4.** *Find an example of a triple $(V, +, \cdot)$ satisfying all of the conditions of the definition of a K-vector space, except for condition (3)(d).*

**Exercise 6.5.** *Let $(V, +, \cdot)$ be a vector space. Show that if $v \in V$ satisfies $v' + v = v'$ for all $v' \in V$, then $v = O$, the additive identity.*

**Exercise 6.6.** *Let $(V, +, \cdot)$ be a vector space. Let $v \in V$. Fix an element $(-v) \in V$ such that $v + (-v) = O$. Suppose that there is $w \in V$ such that $v + w = O$. Show that $w = (-v)$.*

**Exercise 6.7.** *Show the following properties hold for all $v, v_1, v_2 \in V$ and all $\lambda, \lambda_1, \lambda_2 \in K$.*

(1) $0v = O$.
(2) $\lambda O = O$.
(3) $(-\lambda)v = -(\lambda v) = \lambda(-v)$.
(4) *If* $\lambda v = O$, *then either* $\lambda = 0$ *or* $v = O$.
(5) *If* $\lambda v_1 = \lambda v_2$, *then either* $\lambda = 0$ *or* $v_1 = v_2$.
(6) *If* $\lambda_1 v = \lambda_2 v$, *then either* $\lambda_1 = \lambda_2$ *or* $v = O$.
(7) $-(v_1 + v_2) = (-v_1) + (-v_2)$.
(8) $v + v = 2v$, $v + v + v = 3v$, *and in general* $\sum_{i=1}^{n} v = nv$.

exeMAP **Exercise 6.8.** *Consider the set of maps from a set $S$ to $K$. Let us denote this set by $\mathrm{Map}(S, K)$. Define addition and multiplication maps*

$$+ : \mathrm{Map}(S, K) \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

*and*

$$\cdot : K \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

*in the following way. For all $f, g \in \mathrm{Map}(S, K)$, set $f + g$ to be the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$. For all $\lambda \in K$ and all $f \in \mathrm{Map}(S, K)$, set $\lambda \cdot f$ to be the function defined by $(\lambda \cdot f)(x) = \lambda f(x)$ for all $x \in S$.* Show that $(\mathrm{Map}(S, K), +, \cdot)$ is a vector space.

## 2. Sub-$K$-vector spaces

**Definition 6.9** (sub-$K$-vector space)**.** *Let $(V, +, \cdot)$ be a $K$-vector space. A **sub-$K$-vector space** of $(V, +, \cdot)$ is a $K$-vector space $(V', +', \cdot')$ such that $V' \subseteq V$ and such that for all $v', v'_1, v'_2 \in V'$ and all $\lambda \in K$,*

$$v'_1 +' v'_2 = v'_1 + v'_2 \quad \text{and} \quad \lambda \cdot' v' = \lambda \cdot v'.$$

**Definition 6.10.** *If $(V, +, \cdot)$ is a $K$-vector space, and $V' \subseteq V$ is a subset, we say that $V'$ is **closed under $+$ (resp. closed under $\cdot$)** if for all $v'_1, v'_2 \in V'$ (resp. for all $\lambda \in K$ and all $v' \in V'$) we have $v'_1 + v'_2 \in V'$ (resp. $\lambda \cdot v' \in V'$). In this case, we define*

$$+|_{V'} : V' \times V' \to V'$$

*(resp.* $\cdot|_{V'} : K \times V' \to V'$*) to be the map given by* $v_1' +|_{V'} v_2' = v_1' + v_2'$ *(resp.* $\lambda \cdot|_{V'} v' = \lambda \cdot v'$*), for all* $v_1', v_2' \in V'$ *(resp. for all* $\lambda \in K$ *and all* $v' \in V'$*).*

REMARK 6.11. Note that if $(V', +', \cdot')$ is a sub-$K$-vector space of $(V, +, \cdot)$, then $V'$ is closed under $+$ and $\cdot$.

**Exercise 6.12.** *Show that if a non-empty subset* $V' \subseteq V$ *is closed under* $+$ *and* $\cdot$*, then* $(V', +|_{V'}, \cdot|_{V'})$ *is a sub-K-vector space of* $(V, +, \cdot)$.

**Exercise 6.13.** *Show that if* $(V', +', \cdot')$ *is a sub-K-vector space of a K-vector space* $(V, +, \cdot)$*, then the additive identity element* $O' \in V'$ *is equal to the additive identity element* $O \in V$.

**Exercise 6.14.** *Recall the* $\mathbb{R}$*-vector space* $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$ *from Exercise 6.8. In this exercise, show that the subsets of* $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ *listed below are closed under* $+$ *and* $\cdot$*, and so define sub-K-vector spaces of* $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$.

(1) *The set of all polynomial functions.*
(2) *The set of all polynomial functions of degree less than* $n$.
(3) *The set of all functions that are continuos on an interval* $(a, b) \subseteq \mathbb{R}$.
(4) *The set of all functions differentiable at a point* $a \in \mathbb{R}$.
(5) *The set of all functions differentiable on an interval* $(a, b) \subseteq \mathbb{R}$.
(6) *The set of all functions with* $f(1) = 0$.
(7) *The set of all solutions to the differential equation* $f'' + af' + bf = 0$ *for some* $a, b \in \mathbb{R}$.

**Exercise 6.15.** *In this exercise, show that the subsets of* $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ *listed below are NOT closed under* $+$ *and* $\cdot$*, and so do not define sub-K-vector spaces of* $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$.

(1) *Fix* $a \in \mathbb{R}$ *with* $a \neq 0$. *The set of all functions with* $f(1) = a$.
(2) *The set of all solutions to the differential equation* $f'' + af' + bf = c$ *for some* $a, b, c \in \mathbb{R}$ *with* $c \neq 0$.

## 3. Linear maps

**Definition 6.16** (Linear map)**.** *Let* $(V, +, \cdot)$ *and* $(V', +', \cdot')$ *be K-vector spaces. A* ***linear map*** $F : (V, +, \cdot) \to (V', +', \cdot')$ *is a map of sets*

$$f : V \to V'$$

*such that for all* $\lambda \in K$ *and* $v, v_1, v_2 \in V$,

$$f(v_1 + v_2) = f(v_1) +' f(v_2) \quad \textit{and} \quad f(\lambda \cdot v) = \lambda \cdot' f(v).$$

Note that we will frequently use the same letter for the linear map and the map of sets. The $K$-vector space $(V, +, \cdot)$ is called the **source** (or domain) of the linear map and the $K$-vector space $(V', +', \cdot')$ is called the **target** (or codomain) of the linear map. The set $f(V) \subseteq V'$ is called the **image** (or range) of $f$.

**Exercise 6.17.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. Show that the image of* $f$ *is closed under* $+', \cdot'$*, and so defines a sub-K-vector space of the target* $(V', +', \cdot')$.

**Exercise 6.18.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. Show that* $f(O) = O'$.

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**Exercise 6.19.** *Show that the following maps of sets define linear maps of the K-vector spaces.*

(1) *Let $(V, +, \cdot)$ be a K-vector space. Show that the identity map $f : V \to V$, given by $f(v) = v$ for all $v \in V$, is a linear map. This linear map will frequently be denoted by* $\mathrm{Id}_V$.
(2) *Let $(V, +, \cdot)$ and $(V', +', \cdot')$ be K-vector spaces. Show that the zero map $f : V \to V'$, given by $f(v) = O'$ for all $v \in V$, is a linear map.*
(3) *Let $(V, +, \cdot)$ be a K-vector space and let $\alpha \in K$. Show that the multiplication map $f : V \to V$ given by $f(v) = \alpha \cdot v$ for all $v \in V$ is a linear map. This linear map will frequently be denoted by $\alpha \, \mathrm{Id}_V$.*
(4) *Let $a_{ij} \in K$ for $1 \le i \le m$ and $1 \le j \le n$. Show that the map $f : K^n \to K^m$ given by*

$$f(x_1, \ldots, x_n) = \left( \sum_{j=1}^{n} a_{1j} x_j, \ldots, \sum_{j=1}^{n} a_{ij} x_j, \ldots, \sum_{j=1}^{n} a_{mj} x_j \right)$$

*is a linear map.*
(5) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all differentiable real functions $g : \mathbb{R} \to \mathbb{R}$. Let $(V', +', \cdot')$ be the K-vector space of all real functions $g : \mathbb{R} \to \mathbb{R}$. Show that the map $f : (V, +, \cdot) \to (V', +', \cdot')$ that sends a differentiable function $g$ to its derivative $g'$ is a linear map.*
(6) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all continuous real functions $f : \mathbb{R} \to \mathbb{R}$. Show that the map $f : (V, +, \cdot) \to (V, +, \cdot)$ that sends a function $g \in V$ to the function $f(g) \in V$ determined by*

$$f(g)(x) := \int_a^x g(t)dt \quad \text{for all } x \in \mathbb{R}$$

*is a linear map. Make sure to show that $f(g) \in V$ for all $g \in V$.*

**Definition 6.20** (Kernel). *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. The **kernel of** $f$ (or Null space of $f$), denoted $\ker(f)$ (or $\mathrm{Null}(f)$), is the set*

$$\ker(f) := f^{-1}(O') = \{v \in V : f(v) = O'\}.$$

**Exercise 6.21.** *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of K-vector spaces. Show that $\ker(f)$ is a sub-K-vector space of $(V, +, \cdot)$.*

**Exercise 6.22.** *Find the kernel of each of the linear maps listed below (see Problem 6.19).*

(1) *The linear map $\mathrm{Id}_V$.*
(2) *The zero map $V \to V'$.*
(3) *The linear map $\alpha \, \mathrm{Id}_V$.*
(4) *Let $a_{ij} \in K$ for $1 \le i \le m$ and $1 \le j \le n$. The linear map $f : K^n \to K^m$ defined by*

$$f(x_1, \ldots, x_n) = \left( \sum_{j=1}^{n} a_{1j} x_j, \ldots, \sum_{j=1}^{n} a_{ij} x_j, \ldots, \sum_{j=1}^{n} a_{mj} x_j \right).$$

(5) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all differentiable real functions $g : \mathbb{R} \to \mathbb{R}$. Let $(V', +', \cdot')$ be the K-vector space of all real functions $g : \mathbb{R} \to \mathbb{R}$. The linear map $f : (V, +, \cdot) \to (V', +', \cdot')$ that sends a differentiable function $g$ to its derivative $g'$.*
(6) *Let $(V, +, \cdot)$ be the $\mathbb{R}$-vector space of all continous real functions $g : \mathbb{R} \to \mathbb{R}$. The linear map $f : (V, +, \cdot) \to (V, +, \cdot)$ that sends a function $g \in V$ to the*

*function* $f(g) \in V$ *determined by*

$$f(g)(x) := \int_a^x g(t)dt \quad \textit{for all } x \in \mathbb{R}.$$

**Exercise 6.23.** *Show that the composition of linear maps is a linear map.*

**Definition 6.24** (Isomorphism)**.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. We say that* $f$ *is an isomorphism of K-vector spaces if there is a linear map* $g : (V', +', \cdot') \to (V, +, \cdot)$ *of K-vector spaces such that*

$$g \circ f = \mathrm{Id}_{(V,+,\cdot)} \quad \textit{and} \quad f \circ g = \mathrm{Id}_{(V',+',\cdot')} .$$

**Exercise 6.25.** *Show that a linear map is an isomorphism if and only if it is bijective.*