HOMEWORK EXAMPLE

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1. Exercises 18

Exercise 1 (# 18.42). Show that the unity element in a subfield of a field must be the unity element of the whole field (in contrast to Exercise 18.32 for rings in general).

Proof. A field is an integral domain, and since the statement holds for integral domains as well, we will prove this more general fact.

Let D be an integral domain, and let $D' \subseteq D$ be a subring that is also an integral domain. Let $1_{D'}$ to be the multiplicative identity of D' and let 1_D be the multiplicative identity of D. We will show these are equal.

We recall now for later reference that $0_{D'} = 0_D$ (since the identity element of a subgroup is the identity element of the group).

Now let $a \in D'$ be any non-zero element (such an element exists since $1_{D'} \neq 0_{D'}$ by the definition of an integral domain). We have in D' that

$$a \cdot 1_{D'} = a.$$

Multiplication in D' is induced by that of D, and so this equality also holds in D. Similarly, in D we have

$$a \cdot 1_D = a$$

so that

$$a \cdot 1_{D'} = a \cdot 1_D.$$

This implies

$$a \cdot (1_{D'} - 1_D) = 0_D.$$

Since we are in an integral domain, and by assumption $a \neq 0'_D (= 0_D)$, it follows that

$$1_{D'} - 1_D = 0_D.$$

Thus we have shown $1_{D'} = 1_D$. □

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