Exercises

Exercise 1.48. *Let* $A = \{1, 2, 4, 6\}$, $B = \{3, 2, 5\}$ *and* $C = \{2, 5, 10\}$. *Find the following sets:*

(1) $A \cup B$.
(2) $A \cap B$.
(3) $A - B$.
(4) $B - A$.
(5) $(B \cup C) - A$.
(6) $(A \cup C) \cap B$.
(7) $\mathscr{P}(B)$.

Exercise 1.49. *Let* A, B, C *be sets. Show the following, first using Venn diagrams, and then with a more careful proof.*

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(3) $A - (B \cap C) = (A - B) \cup (A - C)$.
(4) $A - (B \cup C) = (A - B) \cap (A - C)$.
(5) *If* $B, C \subseteq A$, *show* $(B \cap C)^C = B^C \cup C^C$.
(6) *If* $B, C \subseteq A$, *show* $(B \cup C)^C = B^C \cap C^C$.

Exercise 1.50. *Let* J *and* A *be sets. For each* $j \in J$, *let* B_j *be a set. Show the following:*

(1) $A \cup \left(\bigcap_{j \in J} B_j \right) = \bigcap_{j \in J} (A \cup B_j)$.

(2) $A \cap \left(\bigcup_{j \in J} B_j \right) = \bigcup_{j \in J} (A \cap B_j)$.

(3) $A - \left(\bigcap_{j \in J} B_j \right) = \bigcup_{j \in J} (A - B_j)$.

(4) $A - \left(\bigcup_{j \in J} B_j \right) = \bigcap_{j \in J} (A - B_j)$.

(5) *If* $B_j \subseteq A$ *for all* j, *then show* $\left(\bigcap_{j \in J} B_j \right)^C = \bigcup_{j \in J} B_j^C$.

(6) *If* $B_j \subseteq A$ *for all* j, *then show* $\left(\bigcup_{j \in J} B_j \right)^C = \bigcap_{j \in J} B_j^C$.

Exercise 1.51. *Let* A *and* B *be sets. Show that if* $a_1, a_2 \in A$ *and* $b_1, b_2 \in B$, *then* $(a_1, b_1) = (a_2, b_2)$ *if and only if* $a_1 = a_2$ *and* $b_1 = b_2$.

Exercise 1.52. *For each set* u, *the expression* $u \neq u$ *is a statement. Call this statement*

$$p(u) : u \neq u.$$

Use this and the separation condition for sets to show that if there exists a set S, *then there exists an empty set* $\emptyset$.

Exercise 1.53. *Let* X *be a set of sets. Show that there is a set*

$$\bigcup X$$

so that

$$u \in \bigcup X \iff \exists z\, (z \in X \,\wedge\, u \in z);$$

in other words $u \in \bigcup X$ *if and only if* u *is an element of an element of* X.

Exercise 1.54. *Let* X *be a set of sets. We saw above that there is a set*

$$\bigcup X$$

so that

$$u \in \bigcup X \iff \exists z (z \in X \wedge u \in z).$$

Show that if A *and* B *are sets, then*

$$A \cup B = \bigcup \{A, B\}.$$

More generally, show that if A_i *are a collection of sets indexed by a set* I*, then*

$$\bigcup_{i \in I} A_i = \bigcup \{A_i : i \in I\}.$$

Exercise 1.55. *Let* A, B, C *be sets, and let* $f : A \to B$ *and* $g : B \to C$ *be maps. Show that the set*

$$\Gamma_{g \circ f} := \{(a, c) \in A \times C : \text{there exists } b \in B \text{ such that } (a, b) \in \Gamma_f, (b, c) \in \Gamma_g\}.$$

defines a map $g \circ f : A \to C$. *Show also that* $(g \circ f)(a) = g(f(a))$.

Exercise 1.56. *Given a set* A*, there is an identity map* $\mathrm{Id}_A : A \to A$ *sending* $a \mapsto a$. *We will also use the notation* 1_A *for the identity map.*

Exercise 1.57. *Show that a map* $f : A \to B$ *is a bijection if and only if there exists a map* $g : B \to A$ *such that* $g \circ f = \mathrm{Id}_A$ *and* $f \circ g = \mathrm{Id}_B$.

Exercise 1.58. *Show that if a map* $f : A \to B$ *admits a section, then* f *is a surjective map.*

Exercise 1.59. *Suppose that* $f : A \to B$ *is a map of sets, and let* $C \subseteq A$. *True or False. If the statement is true, give a proof. If the statement is false, provide a counter example.*

(a) *True or false:* $f(A - C) \subseteq f(A) - f(C)$.
(b) *True or false:* $f(A) - f(C) \subseteq f(A - C)$.
(c) *True or false: If* f *is injective, then* $f(A - C) = f(A) - f(C)$.
(d) *True or false: If* f *is bijective, then* $f(A - C) = B - f(C)$.

Exercise 1.60. *Given an equivalence relation* $\sim$ *on a set* A*, show that the equivalence classes of elements in* A *give a partition of* A. *Conversely, given a set* A *and a partition* $A = \bigsqcup_{i \in I} A_i$*, show that the rule* $x \sim y$ *if and only if* $x, y \in A_i$ *for some* $i \in I$ *defines an equivalence relation on* A. *Show this defines a bijection between the set of equivalence relations on* A *and the set of partitions of* A.

Exercise 1.61. *Define a relation on* $\mathbb{N} \times \mathbb{N}$ *by the rule that for all* $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$,

$$(a, b) \sim (c, d) \iff a + d = b + c.$$

(1) *Show that* $\sim$ *is an equivalence relation.*
(2) *Show that if* $(a, b) \sim (a', b')$ *and* $(c, d) \sim (c', d')$ *then*

$$(a + c, b + d) \sim (a' + c', b' + d').$$

(3) *Show that if* $(a, b) \sim (a', b')$ *and* $(c, d) \sim (c', d')$ *then*

$$(ac + bd, bc + ad) \sim (a'c' + b'd', b'c' + a'd').$$

(4) *Let* $Z = (\mathbb{N} \times \mathbb{N})/\sim$. *Show that there is a map*

$$+ : Z \times Z \to Z$$

defined by $[(a, b)] + [(c, d)] = [(a + c, b + d)]$.

(5) *Let* $Z = (\mathbb{N} \times \mathbb{N})/\sim$. *Show that there is a map*

$$\cdot : Z \times Z \to Z$$

defined by $[(a, b)] \cdot [(c, d)] = [(ac + bd, bc + ad)]$.

(6) *Let* $0_Z := [(1, 1)]$. *Show that for all* $z \in Z$, $0_Z + z = z$.

(7) *For all* $z \in Z$, *show that there exists* $z' \in Z$ *such that* $z' + z = 0_Z$.

(8) *For all* $x, y, z \in Z$, *show that* $(x + y) + z = x + (y + z)$.

(9) *For all* $x, y \in Z$, *show that* $x + y = y + x$.

(10) *Let* $1_Z = [(1, 0)]$. *Show that for all* $z \in Z$, $1_Z \cdot z = z$.

(11) *For all* $x, y \in Z$, *show that* $x \cdot y = y \cdot x$.

(12) *For all* $x, y, z \in Z$, *show that* $x \cdot (y + z) = x \cdot y + x \cdot z$.