CHAPTER 6

Vector spaces and linear maps

In what follows, fix $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. More generally, K can be any field.

1. Vector spaces

Motivated by our intuition of adding and scaling vectors in the plane (see Figure 1), we make the following definition:

Definition 6.1.1. *A K-**vector space** consists of a triple $(V, +, \cdot)$, where V is a set, and $+ : V \times V \to V$ and $\cdot : K \times V \to V$ are maps, satisfying the following properties:*

(1) *(Group laws)*
   (a) *(Additive identity) There exists an element $\mathscr{O} \in V$ such that for all $v \in V$, $v + \mathscr{O} = v$;*
   (b) *(Additive inverse) For each $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = \mathscr{O}$;*
   (c) *(Associativity of addition) For all $v_1, v_2, v_3 \in V$,*
$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3);$$
(2) *(Abelian property)*
   (a) *(Commutativity of addition) For all $v_1, v_2 \in V$,*
$$v_1 + v_2 = v_2 + v_1;$$
(3) *(Module conditions)*
   (a) *For all $\lambda \in K$ and all $v_1, v_2 \in V$,*
$$\lambda \cdot (v_1 + v_2) = (\lambda \cdot v_1) + (\lambda \cdot v_2);$$
   (b) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 + \lambda_2) \cdot v = (\lambda_1 \cdot v) + (\lambda_2 \cdot v);$$
   (c) *For all $\lambda_1, \lambda_2 \in K$, and all $v \in V$,*
$$(\lambda_1 \lambda_2) \cdot v = \lambda_1 \cdot (\lambda_2 \cdot v);$$
   (d) *For all $v \in V$,*
$$1 \cdot v = v.$$

In the above, for all $\lambda \in K$ and all $v, v_1, v_2 \in V$ we have denoted $+(v_1, v_2)$ by $v_1 + v_2$ and $\cdot(\lambda, v)$ by $\lambda \cdot v$.

In addition, for brevity, we will often write λv for $\lambda \cdot v$.

EXAMPLE 6.1.2 (The vector space K^n). By definition,
$$K^n = \{(x_1, \ldots, x_n) \ : \ x_i \in K, \ 1 \le i \le n\}.$$
The map $+ : K^n \times K^n \to K^n$ is defined by the rule
$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

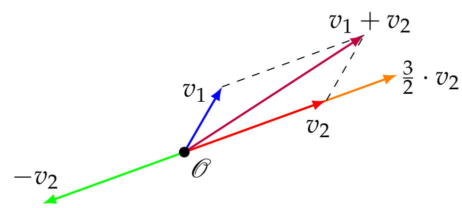

FIGURE 1. Adding and scaling vectors in the plane

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in K^n$. The map $\cdot : K \times K^n \to K^n$ is defined by the rule

$$\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

for all $\lambda \in K$ and $(x_1, \ldots, x_n) \in K^n$.

Exercise 6.1.3. *Show that $(K^n, +, \cdot)$, defined in the example above, is a K-vector space.*

Exercise 6.1.4 (Cancelation rule). *Let $(V, +, \cdot)$ be a K-vector space. Show that if we have $v_1, v_2, w \in V$, then*

$$v_1 + w = v_2 + w \iff v_1 = v_2.$$

Exercise 6.1.5 (Unique additive identity). *Let $(V, +, \cdot)$ be a K-vector space. Fix an element $\mathscr{O} \in V$ such that for all $v \in V$, we have $v + \mathscr{O} = v$. Show that if $w \in V$ satisfies $v' + w = v'$ for all $v' \in V$, then $w = \mathscr{O}$.*

Exercise 6.1.6 (Unique additive inverse). *Let $(V, +, \cdot)$ be a K-vector space. Let $v \in V$. Fix an element $-v \in V$ such that $v + (-v) = \mathscr{O}$. Suppose that there is $w \in V$ such that $v + w = \mathscr{O}$. Show that $w = -v$.*

Exercise 6.1.7. *Let $(V, +, \cdot)$ be a K-vector space. Show the following properties hold for all $v, v_1, v_2 \in V$ and all $\lambda, \lambda_1, \lambda_2 \in K$.*

(1) $0v = \mathscr{O}$.
(2) $\lambda \mathscr{O} = \mathscr{O}$.
(3) $(-\lambda)v = -(\lambda v) = \lambda(-v)$.
(4) *If $\lambda v = \mathscr{O}$, then either $\lambda = 0$ or $v = \mathscr{O}$.*
(5) *If $\lambda v_1 = \lambda v_2$, then either $\lambda = 0$ or $v_1 = v_2$.*
(6) *If $\lambda_1 v = \lambda_2 v$, then either $\lambda_1 = \lambda_2$ or $v = \mathscr{O}$.*
(7) $-(v_1 + v_2) = (-v_1) + (-v_2)$.
(8) *$v + v = 2v$, $v + v + v = 3v$, and in general $\sum_{i=1}^{n} v = nv$.*

Exercise 6.1.8. *Consider the set of maps from a set S to K. Let us denote this set by* $\mathrm{Map}(S, K)$. *Define addition and multiplication maps*

$$+ : \mathrm{Map}(S, K) \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

and

$$\cdot : K \times \mathrm{Map}(S, K) \to \mathrm{Map}(S, K)$$

in the following way. For all $f, g \in \mathrm{Map}(S, K)$, set $f + g$ to be the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$. For all $\lambda \in K$ and all $f \in \mathrm{Map}(S, K)$, set $\lambda \cdot f$ to be the function defined by $(\lambda \cdot f)(x) = \lambda f(x)$ for all $x \in S$. Show that if $S \neq \emptyset$ then $(\mathrm{Map}(S, K), +, \cdot)$ is a K-vector space.

2. Sub-K-vector spaces

Definition 6.2.9 (sub-K-vector space)**.** *Let $(V,+,\cdot)$ be a K-vector space. A **sub-K-vector space** of $(V,+,\cdot)$ is a K-vector space $(V',+',\cdot')$ such that $V' \subseteq V$ and such that for all $v', v_1', v_2' \in V'$ and all $\lambda \in K$,*

$$v_1' +' v_2' = v_1' + v_2' \quad \text{and} \quad \lambda \cdot' v' = \lambda \cdot v'.$$

We will write $(V',+',\cdot') \subseteq (V,+,\cdot)$.

Definition 6.2.10. *If $(V,+,\cdot)$ is a K-vector space, and $V' \subseteq V$ is a subset, we say that V' is **closed under $+$ (resp. closed under $\cdot$)** if for all $v_1', v_2' \in V'$ (resp. for all $\lambda \in K$ and all $v' \in V'$) we have $v_1' + v_2' \in V'$ (resp. $\lambda \cdot v' \in V'$). In this case, we define*

$$+|_{V'} : V' \times V' \to V'$$

(resp. $\cdot|_{V'} : K \times V' \to V'$) to be the map given by $v_1' +|_{V'} v_2' = v_1' + v_2'$ (resp. $\lambda \cdot|_{V'} v' = \lambda \cdot v'$), for all $v_1', v_2' \in V'$ (resp. for all $\lambda \in K$ and all $v' \in V'$).

REMARK 6.2.11. Note that if $(V',+',\cdot')$ is a sub-K-vector space of $(V,+,\cdot)$, then V' is closed under $+$ and $\cdot$.

Exercise 6.2.12. *Show that if a non-empty subset $V' \subseteq V$ is closed under $+$ and $\cdot$, then $(V', +|_{V'}, \cdot|_{V'})$ is a sub-K-vector space of $(V,+,\cdot)$.*

Exercise 6.2.13. *Show that if $(V',+',\cdot')$ is a sub-K-vector space of a K-vector space $(V,+,\cdot)$, then the additive identity element $\mathscr{O}' \in V'$ is equal to the additive identity element $\mathscr{O} \in V$.*

Exercise 6.2.14. *Recall the $\mathbb{R}$-vector space $(\mathrm{Map}(\mathbb{R},\mathbb{R}),+,\cdot)$ from Exercise 6.1.8. In this exercise, show that the subsets of $\mathrm{Map}(\mathbb{R},\mathbb{R})$ listed below are closed under $+$ and $\cdot$, and so define sub-$\mathbb{R}$-vector spaces of $(\mathrm{Map}(\mathbb{R},\mathbb{R}),+,\cdot)$.*

(1) *The set of all polynomial functions.*
(2) *The set of all polynomial functions of degree less than n.*
(3) *The set of all functions that are continuos on an interval $(a,b) \subseteq \mathbb{R}$.*
(4) *The set of all functions differentiable at a point $a \in \mathbb{R}$.*
(5) *The set of all functions differentiable on an interval $(a,b) \subseteq \mathbb{R}$.*
(6) *The set of all functions with $f(1) = 0$.*
(7) *The set of all solutions to the differential equation $f'' + af' + bf = 0$ for some $a, b \in \mathbb{R}$.*

Exercise 6.2.15. *In this exercise, show that the subsets of $\mathrm{Map}(\mathbb{R},\mathbb{R})$ listed below are NOT closed under $+$ and $\cdot$, and so do not define sub-$\mathbb{R}$-vector spaces of $(\mathrm{Map}(\mathbb{R},\mathbb{R}),+,\cdot)$.*

(1) *Fix $a \in \mathbb{R}$ with $a \neq 0$. The set of all functions with $f(1) = a$.*
(2) *The set of all solutions to the differential equation $f'' + af' + bf = c$ for some $a, b, c \in \mathbb{R}$ with $c \neq 0$.*

3. Linear maps

Definition 6.3.16 (Linear map)**.** *Let $(V,+,\cdot)$ and $(V',+',\cdot')$ be K-vector spaces. A **linear map** $F : (V,+,\cdot) \to (V',+',\cdot')$ is a map of sets*

$$f : V \to V'$$

such that for all $\lambda \in K$ *and* $v, v_1, v_2 \in V$,

$$f(v_1 + v_2) = f(v_1) +' f(v_2) \quad \text{and} \quad f(\lambda \cdot v) = \lambda \cdot' f(v).$$

Note that we will frequently use the same letter for the linear map and the map of sets. The K-vector space $(V, +, \cdot)$ is called the **source** (or domain) of the linear map and the K-vector space $(V', +', \cdot')$ is called the **target** (or codomain) of the linear map. The set $f(V) \subseteq V'$ is called the **image** (or range) of f.

Exercise 6.3.17. *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. Show that the image of* f *is closed under* $+', \cdot'$*, and so defines a sub-K-vector space of the target* $(V', +', \cdot')$.

Exercise 6.3.18. *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. Show that* $f(\mathscr{O}) = \mathscr{O}'$.

Exercise 6.3.19. *Show that the following maps of sets define linear maps of the K-vector spaces.*

(1) *Let* $(V, +, \cdot)$ *be a K-vector space. Show that the identity map* $f : V \to V$*, given by* $f(v) = v$ *for all* $v \in V$*, is a linear map. This linear map will frequently be denoted by* Id_V.
(2) *Let* $(V, +, \cdot)$ *and* $(V', +', \cdot')$ *be K-vector spaces. Show that the zero map* $f : V \to V'$*, given by* $f(v) = \mathscr{O}'$ *for all* $v \in V$*, is a linear map.*
(3) *Let* $(V, +, \cdot)$ *be a K-vector space and let* $\alpha \in K$*. Show that the multiplication map* $f : V \to V$ *given by* $f(v) = \alpha \cdot v$ *for all* $v \in V$ *is a linear map. This linear map will frequently be denoted by* $\alpha\,\mathrm{Id}_V$.
(4) *Let* $a_{ij} \in K$ *for* $1 \le i \le m$ *and* $1 \le j \le n$*. Show that the map* $f : K^n \to K^m$ *given by*

$$f(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j, \ldots, \sum_{j=1}^{n} a_{ij} x_j, \ldots, \sum_{j=1}^{n} a_{mj} x_j \right)$$

is a linear map.
(5) *Let* $(V, +, \cdot)$ *be the* $\mathbb{R}$*-vector space of all differentiable real functions* $g : \mathbb{R} \to \mathbb{R}$*. Let* $(V', +', \cdot')$ *be the* $\mathbb{R}$*-vector space of all real functions* $g : \mathbb{R} \to \mathbb{R}$*. Show that the map* $f : (V, +, \cdot) \to (V', +', \cdot')$ *that sends a differentiable function* g *to its derivative* g' *is an linear map.*
(6) *Let* $(V, +, \cdot)$ *be the* $\mathbb{R}$*-vector space of all continuous real functions* $f : \mathbb{R} \to \mathbb{R}$*. Show that the map* $f : (V, +, \cdot) \to (V, +, \cdot)$ *that sends a function* $g \in V$ *to the function* $f(g) \in V$ *determined by*

$$f(g)(x) := \int_a^x g(t)dt \quad \text{for all } x \in \mathbb{R}$$

is a linear map. Make sure to show that $f(g) \in V$ *for all* $g \in V$.

Definition 6.3.20 (Kernel)**.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. The* ***kernel of*** f *(or Null space of* f*), denoted* $\ker(f)$ *(or* $\mathrm{Null}(f)$*), is the set*

$$\ker(f) := f^{-1}(\mathscr{O}') = \{v \in V : f(v) = \mathscr{O}'\}.$$

Exercise 6.3.21. *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of K-vector spaces. Show that* $\ker(f)$ *is a sub-K-vector space of* $(V, +, \cdot)$.

Exercise 6.3.22. *Find the kernel of each of the linear maps listed below (see Problem 6.3.19).*

(1) *The linear map* Id_V.
(2) *The zero map* $V \to V'$.
(3) *The linear map* $\alpha\,\mathrm{Id}_V$.
(4) *Let* $a_{ij} \in K$ *for* $1 \leq i \leq m$ *and* $1 \leq j \leq n$. *The linear map* $f : K^n \to K^m$ *defined by*
$$f(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j, \ldots, \sum_{j=1}^{n} a_{ij} x_j, \ldots, \sum_{j=1}^{n} a_{mj} x_j \right).$$
(5) *Let* $(V, +, \cdot)$ *be the* $\mathbb{R}$*-vector space of all differentiable real functions* $g : \mathbb{R} \to \mathbb{R}$. *Let* $(V', +', \cdot')$ *be the* $\mathbb{R}$*-vector space of all real functions* $g : \mathbb{R} \to \mathbb{R}$. *The linear map* $f : (V, +, \cdot) \to (V', +', \cdot')$ *that sends a differentiable function* g *to its derivative* g'.
(6) *Let* $(V, +, \cdot)$ *be the* $\mathbb{R}$*-vector space of all continous real functions* $g : \mathbb{R} \to \mathbb{R}$. *Let* $a \in \mathbb{R}$. *The linear map* $f : (V, +, \cdot) \to (V, +, \cdot)$ *that sends a function* $g \in V$ *to the function* $f(g) \in V$ *determined by*
$$f(g)(x) := \int_a^x g(t)dt \quad \textit{for all } x \in \mathbb{R}.$$

Exercise 6.3.23. *Show that the composition of linear maps is a linear map.*

Definition 6.3.24 (Isomorphism)**.** *Let* $f : (V, +, \cdot) \to (V', +', \cdot')$ *be a linear map of* K*-vector spaces. We say that* f *is an isomorphism of* K*-vector spaces if there is a linear map* $g : (V', +', \cdot') \to (V, +, \cdot)$ *of* K*-vector spaces such that*
$$g \circ f = \mathrm{Id}_{(V,+,\cdot)} \quad \textit{and} \quad f \circ g = \mathrm{Id}_{(V',+',\cdot')}.$$

Exercise 6.3.25. *Show that a linear map is an isomorphism if and only if it is bijective.*