# PRACTICE FINAL

MATH 3140

1:00 PM Wednesday April 27, 2011 to 1:00 PM Friday April 29, 2011

Name | ______________________________

Please answer the all of the questions, and show your work. You must **hand your exam to me in person**, in class on Friday (do not leave your exam in a mailbox or under my door). You may consult your textbook, your class notes, your homework, your exams, the three practice exams, and **nothing else**. Do not discuss the exam with anyone except for me.

| 1 | 2 | 3 | 4 | 5 | 6 | | |
|---|---|---|---|---|---|---|---|
| 10 | 10 | 10 | 10 | 10 | 10 | total | percent |

*Date*: April 24, 2011.

| 1 |
| --- |
| 10 points |

1. Let $\mathbb{Q}[i] = \{a + bi : a, b \in \mathbb{Q}\} \subseteq \mathbb{C}$.

1(a) [3 points]. Show that $\mathbb{Q}[i]$ is a subfield of $\mathbb{C}$.

1(b) [3 points]. Show that $(x^2 + 1) := \{(x^2 + 1)g(x) : g(x) \in \mathbb{Q}[x]\}$ is an ideal in $\mathbb{Q}[x]$.

1(c) [4 points]. It is a fact that any ideal $I$ in $\mathbb{Q}[x]$ such that $(x^2 + 1) \subseteq I \subseteq \mathbb{Q}[x]$ is either equal to $(x^2 + 1)$ or $\mathbb{Q}[x]$. Use this to show that $\mathbb{Q}[i]$ is isomorphic to the quotient ring $\mathbb{Q}[x]/(x^2 + 1)$. [Hint: consider an evaluation homomorphism.]

| 2 |
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| 10 points |

2(a) [2 points]. Let $R$ be a ring with unity $1_R$, let $R'$ be a ring with no zero divisors, and let $\phi : R \to R'$ be a non-zero homomorphism. Show that $R'$ has a multiplicative identity element equal to $\phi(1_R)$.

2(b) [4 points]. Find all ring homomorphisms from $\mathbb{Z}_p$ to $\mathbb{Z}_p$.

2(c) [4 points]. Find all ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$.

| 3 |
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| 10 points |

3(a) [2 points]. In a commutative ring with unity, show that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ for all $a, b$ in the ring. [Hint: First show that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$, then use induction.]

3(b) [8 points]. An element $r$ of a ring $R$ is said to be nilpotent if there exists some $n \in \mathbb{N}$ such that $r^n = 0$. Let $N$ be the set of nilpotent elements of a commutative ring $R$ with unity. Show that $N$ is an ideal in $R$.

| 4 |
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| 10 points |

4. Show that for a prime $p$, $x^p + a \in \mathbb{Z}_p[x]$ is not irreducible for any $a \in \mathbb{Z}_p$.

| 5 |
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| 10 points |

5. Show that a finite, simple, abelian group has prime order. [Hint: use the Fundamental Theorem of Finitely Generated Abelian Groups.]

| 6 |
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| 10 points |

6. True or false.

6(a). A quotient ring of an integral domain is an integral domain.

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6(b). Every quotient group of a cyclic group is cyclic.

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6(c). Let $n \in \mathbb{N}$. There is a single group $G$ of order $n!$ such that any finite group of order $n$ is isomorphic to a subgroup of $G$.

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6(d). Let $p$ and $q$ be primes. A proper subgroup of a group of order $pq$ is cyclic.

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6(e). The characteristic of a ring is a prime number.

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6(f). The direct product of two fields is a field.

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6(g). For a prime $p$, and an integer $z$, we have $z^p \equiv z \pmod p$.

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6(h). If $R$ is a ring, then the zero divisors of $R[x]$ are precisely the zero divisors of $R$.

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6(i). The polynomial $x^7 - 2$ is irreducible over $\mathbb{Q}$.

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6(j). If $F$ is a field, then there exist irreducible polynomials in $F[x]$ of every positive degree.

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