MATH4450: HOMEWORK 9

DUE FRIDAY OCTOBER 31

1. Exercises

1.1. **Exercise (3) (a). Prove the following theorem** using the outline below.

Theorem 1.1. *Let $f : U \to \mathbb{R}$ be a continuous function on an open set $U \subset \mathbb{R}^2$. Fixing coordinates x and y on $\mathbb{R}^2$, suppose further that $\partial f/\partial x : U \to \mathbb{R}$ exists and is also continuous. Then given a closed set $V = [x_0, x_1] \times [y_0, y_1] \subseteq U$, for any $x \in [x_0, x_1]$ we have*

$$\frac{d}{dx}\int_{y_0}^{y_1} f(x,y)dy = \int_{y_0}^{y_1} \frac{\partial f(x,y)}{\partial x}dy.$$

[Hint: use the fact from multivariable calculus that if

$$f : [x_0, x_1] \times [y_0, y_1] \to \mathbb{R}$$

is continuous, then

$$\int_{y_0}^{y_1}\left(\int_{x_0}^{x_1} f(x,y)dx\right)dy = \int_{x_0}^{x_1}\left(\int_{y_0}^{y_1} f(x,y)dy\right)dx,$$

and rewrite the latter integral in the theorem as

$$\int_{y_0}^{y_1} \frac{\partial f(x,y)}{\partial x}dy = \frac{d}{dx}\int_{x_0}^{x}\left(\int_{y_0}^{y_1}\frac{\partial f(t,y)}{\partial t}dy\right)dt.]$$

1.2. **Exercise (3) (b).** WARNING: It is tempting to write

$$\begin{aligned}
\frac{d}{dx}\int_{y_0}^{y_1} f(x,y)dy &= \lim_{h\to 0}\frac{1}{h}\left(\int_{y_0}^{y_1} f(x+h,y) - \int_{y_0}^{y_1} f(x,y)dy\right)\\
&= \lim_{h\to 0}\left(\int_{y_0}^{y_1}\frac{\partial f(t,y)}{\partial t}dy + \int_{y_0}^{y_1} r(x,y,h)dz\right)\\
&= \int_{y_0}^{y_1}\frac{\partial f(t,y)}{\partial t}dy + \lim_{h\to 0}\int_{y_0}^{y_1} r(x,y,h)dz
\end{aligned}$$

where

$$r(x,y,h) = \frac{f(x+h,y) - f(x,y)}{h} - \frac{\partial f(t,y)}{\partial t}$$

and
$$\lim_{h\to 0} r(x,y,h) = 0.$$
But recall, and this is the warning, *it is not always the case that if a function converges to zero, that its integral converges to zero.* [One needs a stronger assumption such as uniform convergence.]

As an exercise, **prove the following proposition**:

Proposition 1.2. *There exists a continuous function* $f : (0,1]\times\mathbb{R} \to \mathbb{R}$ *such that for each fixed* $x \in \mathbb{R}$, $\lim_{h\to 0^+} f(h,x) = 0$, *but*
$$\lim_{h\to 0^+} \int_0^2 f(h,x)dx \neq 0.$$

[Hint: Consider the function
$$g(h,x) = \begin{cases} 0 & \text{if } x \le 0 \\ hx & \text{if } 0 \le x \le 1/h \\ 2 - hx & \text{if } 1/h \le x \le 2/h \\ 0 & \text{if } 2/h \ge x \end{cases}$$
and set $f(h,x) = hg(h,x)$.]