MATH 232: HOMEWORK 1

DUE WEDNESDAY FEBRUARY 20

Problems

Let X be a smooth projective variety of dimension n over the complex numbers.

(1) Given a Cartier divisor D on X, let L_D be the associated line bundle. Show that the map

$$\mathrm{Cl}(X) \to \mathrm{Pic}(X)$$

given by

$$D \mapsto L_D$$

induces an isomorphism from the group of divisors on X, modulo linear equivalence, to the group of line bundles on X.

(2) Let D be a divisor on X, with associated line bundle L_D, and let $|D|$ be the associated linear series

$$|D| = \{E \in \mathrm{Div}(X) : E \geq 0, E \sim D\}$$

Show there is a bijection between $|D|$ and $\mathbb{P}\Gamma(X, L_D)$.

(3) Let $D \geq 0$ be an effective divisor on X, and let L_D be the associated line bundle, with sheaf of sections $\mathcal{O}_X(D)$. Fix a section $s \in \Gamma(X, L_D)$ which vanishes along D.

(a) Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X over which $L_D^\vee$ is trivialized. On each open chart U_α let s_α be the corresponding section $s_\alpha : U_\alpha \to \mathbb{C}$.
Show that s induces a well defined map

$$L_D^\vee \xrightarrow{s} X \times \mathbb{C}$$

given locally on an open chart

$$U_\alpha \times \mathbb{C} \to U_\alpha \times \mathbb{C}$$

by

$$(p, z) \mapsto (p, s_\alpha(p) z).$$

(b) Show that this is not a morphism of line bundles in the sense that the map is not of constant rank.

(c) Show that this map nevertheless induces an injection of locally free sheaves of rank one

$$0 \to \mathscr{O}_X(-D) \xrightarrow{s} \mathscr{O}_X,$$

and that moreover, the image of $\mathscr{O}_X(-D)$ in $\mathcal{O}_X$ via this map is the sheaf of regular functions on X vanishing along D.

(4) Let $p \in X$ be a point, and let $\pi : Y \to X$ be the blow-up of X at p, with exceptional divisor E. For $n > 0$, show that

$$K_Y \cong \pi^* K_X + (n-1)E.$$

(5) Recall that for an irreducible curve C, the arithmetic genus of C is defined as $p_a(C) = h^1(C, \mathscr{O}_C)$. If C lies on a smooth surface X, show that

$$2p_a(C) - 2 = C(C + K_X).$$

(6) Again let C be an irreducible curve on a smooth surface X. Show that there is a morphism $Y \to X$ consisting of a finite number of blow-ups, such that the strict transform of C is smooth. [Hint: show that blowing up a singular point of C strictly decreases the arithmetic genus of C.]