SAMPLE MIDTERM I
ANALYSIS 1

MATH 3100

Friday October 4, 2013

Name | ______________________________

Please answer all of the questions, and show your work.
All solutions must be explained clearly to receive credit.

1	2	3	4	5	6	7	
10	10	10	10	10	10	10	Total

Date: September 29, 2013.

1
10 points

1. Let S and T be non-empty sets. Let $\mathrm{Map}(S, T)$ be the set of maps from S to T.

1.(a). What is the definition of a map from S to T.

1.(b). If T is uncountable, show that $\mathrm{Map}(S, T)$ is uncountable.

1.(c). If S is infinite and T has at least 2 elements, show that $\mathrm{Map}(S, T)$ uncountable.

2
10 points

2. Let K be a field, and let $\leq$ be a total ordering on K.

2.(a). What conditions on $\leq$ must be satisfied for $(K, \leq)$ to be an ordered field.

2.(b). Assume $(K, \leq)$ is an odered field. Use the definition of an ordered field to show that if $x_1, x_2, x_3 \in K$, $x_1 > x_2$ and $x_3 < 0$, then $x_1 x_3 < x_1 x_2$. (You may use facts about fields (that are not ordered) without proof.)

2.(c). Assume $(K, \leq)$ is an ordered field. Use Part (b) to show that $x^2 \geq 0$ for all $x \in K$. (You may use facts about fields (that are not ordered) without proof.)

3
10 points

3. Let $(K, \leq)$ be an ordered field.

3.(a). One can show (*although you do NOT need to do so here*) that for $a \in K$, $|x| \leq a$ if and only if $-a \leq x \leq a$. Use this to show that for $x_1, x_2 \in K$,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

3.(b). Use part (a) to show by induction that for any $n \in \mathbb{N}$, if $x_1, \ldots, x_n \in K$, then

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|.$$

4
10 points

4. Let $n \in \mathbb{N}$. Suppose that $x, y \in \mathbb{R}$, $x, y \geq 0$. Show *either* of the following (*your choice*):

(1) If $x^n < y$, then there exists $t \in \mathbb{R}$ with $x < t$ such that $x^n < t^n < y$.

(2) If $x^n > y$, then there exists $t \in \mathbb{R}$ with $x > t > 0$ such that $x^n > t^n > y$.

5
10 points

5. Consider the set

$$S = \left(\bigcup_{n \in \mathbb{N}} \{1/n\} \right) \cup (2, 3]$$

Answer the following questions. *You do NOT need to prove your answers.*

5.(a). What is $\sup S$, the supremum of S (if it exists)?

5.(b). What is $\inf S$, the infemum of S (if it exists)?

5.(c). What is S°, the interior of S?

5.(d). What is $\overline{S}$, the closure of S?

5.(e). What is ∂S, the boundary of S?

5.(f). What is $L(S)$, the set of limit points of S?

5.(g). What is the set of isolated points of S?

6
10 points

6.(a). Show that if a subset $S \subseteq \mathbb{R}$ of the reals is not an interval, then there are open subsets $U \subseteq \mathbb{R}$ and $U' \subseteq \mathbb{R}$ with $U \cap U' = \emptyset$, $U \cap S \neq \emptyset$, and $U' \cap S \neq \emptyset$, such that $S \subseteq U \cup U'$.

6.(b). Show that if a subset $S \subseteq \mathbb{R}$ of the reals is an interval, then there do not exist open subsets $U \subseteq \mathbb{R}$ and $U' \subseteq \mathbb{R}$ with $U \cap U' = \emptyset$, $U \cap S \neq \emptyset$, and $U' \cap S \neq \emptyset$, such that $S \subseteq U \cup U'$. [Hint: Let $a \in U \cap S$ and $b \in U' \cap S$. Assume WLOG that $a < b$. Then set $T = \{t \in [a, b] : t \in U\}$.]

7
10 points

7. True or False. *You do NOT need to justify your answer.*

☐ **7.(a)** The set of subsets of a countable set is countable.

☐ **7.(b)** If K is a field, with a total order $<$, and $k^2 < 0$ for some $k \in K$, then K is not an ordered field.

☐ **7.(c)** Every ordered field is Archimedean

☐ **7.(d)** Every non-empty subset of $\mathbb{N}$ contains a minimum.

☐ **7.(e)** If x and y are irrational then xy is irrational.

☐ **7.(f)** A countable union of countable sets is countable.

☐ **7.(g)** Every compact subset of $\mathbb{R}$ has a limit point.

☐ **7.(h)** The closure of a set is closed.

☐ **7.(i)** Every Archimedean ordered field is complete.

☐ **7.(j)** If $S \subseteq \mathbb{R}$, then the set of limit points $L(S)$ is closed.