ABSTRACT ALGEBRA 2
MATH 4140
LINEAR ALGEBRA HOMEWORK

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1. Exercises

For these exercises, we will assume $k \in \{\mathbb{R}, \mathbb{Q}, \mathbb{C}\}$. Unless otherwise indicated, V will denote a k-vector space.

Exercise 1.1. Show that a linear map of k-vector spaces is an isomorphism if and only if it is both injective and surjective.

Exercise 1.2. Show that a linear map of k-vector spaces is injective if and only if its kernel is trivial (equal to $\{0\}$).

Exercise 1.3. Let V be a k-vector space. Given elements $v_1, \ldots, v_n \in V$, we define a map of sets

$$L = L_{v_1,\ldots,v_n} : k^n \to V$$

by the rule

$$(r_1, \ldots, r_n) \mapsto \sum_{i=1}^{n} r_i v_i.$$

Show that L is a linear map.

Exercise 1.4. Recall that in the notation of the previous problem, the elements $v_1, \ldots, v_n \in V$ are said to be linearly independent if $\ker(L_{v_1,\ldots,v_n}) = \{0\}$. They are said to be a basis of V if $L_{v_1,\ldots,v_n}$ is an isomorphism.

Show that $v_1, \ldots, v_n \in V$ are linearly independent if and only if $\sum_{i=1}^{n} r_i v_i = 0$ implies that $r_1 = \ldots = r_n = 0$.

Exercise 1.5. Show that the set of polynomials in one variable, with coefficients in k (i.e. $k[x]$) is a k-vector space. Show that it is not finite dimensional.

Exercise 1.6. The image of a linear map $f : V \to W$ is the set $f(V)$. Show that the kernel of f is a linear subspace of V and the image of f is a linear subspace of W.

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Exercise 1.7. Suppose $A \in M_{m\times n}(k)$, $B \in M_{n\times p}(k)$ and $A \in M_{p\times q}(k)$. Show that $(AB)C = A(BC) \in M_{m\times q}(k)$.

Exercise 1.8. For $r \in k$, and $A \in M_{m\times n}k)$, define rA by the rule $(rA)_{ij} = r(A_{ij})$. Show that this makes $M_{m\times n}(k)$ into an k-vector space of dimension nm.

Exercise 1.9. For vector spaces V and W, we denote by $\mathrm{Hom}_k(V,W)$ the set of k-linear maps. For $f, g \in \mathrm{Hom}_k(V,W)$, define $f+g$ by $(f+g)(v) = f(v)+g(v)$. For $r \in k$ and $f \in \mathrm{Hom}_k(V,W)$, define rf by $(rf)(v) = r(f(v))$. Show that this makes $\mathrm{Hom}_k(V,W)$ into an k-vector space.

Exercise 1.10. Define $e_i \in k^n = M_{1,n}(k)$ to be the vector that has all zero entries, except for a 1 in the i-th place. Similarly, define $\hat{e}_i \in k^m = M_{1,m}(k)$ to be the vector that has all zero entries, except for a 1 in the i-th place. Define a map

$$M : \mathrm{Hom}_k(k^n, k^m) \to M_{m\times n}(k)$$

by

$$(M(f))_{ij} = \hat{e}_i f(e_j)^T.$$

Show that M is a linear isomorphism.

Exercise 1.11. Show that $\det(A^{-1}) = (\det(A))^{-1}$.

Exercise 1.12. Show $\det A = 0$ if and only if $\ker A \neq \{0\}$.

Exercise 1.13. Let $k^* = \{r \in K : r \neq 0\}$. Denote by $*$ multiplication on k. Show that $(k^*, *)$ is a group.

Exercise 1.14. Show that det induces a surjective homomorphism of groups $GL_n(k) \to k^*$ with kernel equal to $SL_n(k)$.

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