ABSTRACT ALGEBRA 2 PRACTICE EXAM

1. Practice exam problems

Problem A. Find $\alpha \in \mathbb{C}$ such that $\mathbb{Q}(i, \sqrt[3]{2}) = \mathbb{Q}(\alpha)$.

Problem B. Let ϕ_2 be the Frobenius automorphism of $\mathbb{F}_4$, the field with 4 elements. Let $0, 1, \alpha, \beta$ be the elements of $\mathbb{F}_4$. Describe ϕ_2 by indicating the image of each element of $\mathbb{F}_4$ under this map (e.g. $\phi_2(0) = 0$).

Problem C. Give an example of a degree two field extension that is not Galois.

Problem D. Let $\zeta \in \mathbb{C}$ be a primitive 5-th root of unity. Find all field extensions K of $\mathbb{Q}$ contained in $\mathbb{Q}(\zeta)$. For each such field extension, find an element $\alpha \in \mathbb{Q}(\zeta)$ such that $K = \mathbb{Q}(\alpha)$.

Problem E. Let F be a field. For a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in F[x]$ we define the derivative $f'(x)$ of $f(x)$ to be the polynomial

$$f'(x) = \sum_{i=1}^{n} i a_i x^{i-1}.$$

(a) Show that the map $D : F[x] \to F[x]$ given by $D(f(x)) = f'(x)$ is a linear map of vector spaces.
(b) Find $\ker(D)$. [Hint: The answer may depend on the characteristic of F.]
(c) Show that D satisfies the Leibniz rule: $D(f(x)g(x)) = D(f(x))g(x) + f(x)D(g(x))$ for all $f(x), g(x) \in F[x]$.
(d) Show that $D((f(x)^m)) = mf(x)^{m-1}D(f)$ for each $m \in \mathbb{Z}_{\geq 0}$.

Problem F. Let $\bar{F}$ be an algebraic closure of a field F. Show that $f(x) \in F[x]$ has a root $\alpha \in \bar{F}$ of multiplicity $\mu > 1$ if and only if α is a root of both $f(x)$ and $f'(x)$. [Hint: Consider the factorization $f(x) = (x - \alpha)^{\mu} g(x)$ in $\bar{F}[x]$ and use the previous problem.]

Problem G. Let E/F be an extension of fields. Let K_1, K_2 be two finite field extensions of F contained in E. Show that if K_1 is a normal extension of F, then K_1K_2 is a normal extension of K_2.

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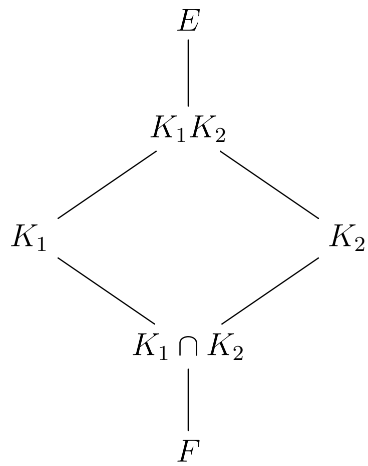
Problem H (Optional)**.** Let E be a finite Galois extension of a field F. Let K_1 and K_2 be two extensions of F contained in E. We obtain a diagram of field extensions



Show that $G(E/(K_1K_2)) = G(E/K_1) \cap G(E/K_2) \subseteq G(E/F)$ and $G(E/(K_1 \cap K_2)$ is the subgroup G of $G(E/F)$ generated by the set

$$G(E/K_1)G(E/K_2) = \{\sigma_1\sigma_2 : \sigma_1 \in G(E/K_1), \sigma_2 \in G(E/K_2)\}.$$

[Hint: For the first part, to show $G(E/(K_1K_2)) \supseteq G(E/K_1) \cap G(E/K_2)$, come up with a useful description of the elements of K_1K_2 in terms of those in K_1 and K_2. For the second part, use Galois theory to show $E^G = K_1 \cap K_2$.]

Problem I. Let E/F be an extension of fields. Let K_1, K_2 be two field extensions of F contained in E. If K_1 is a finite Galois extension of F, then K_1K_2 is Galois over K_2. Moreover, there is an isomorphism

$$\phi : G(K_1K_2/K_2) \to G(K_1/(K_1 \cap K_2))$$

given by $\sigma \mapsto \sigma|_{K_1}$.