MATH 2300 – review problems for Exam 3 ANSWERS

1. Check whether the following series converge or diverge. In each case, justify your answer by either computing the sum or by by showing which convergence test you are using, why and how it applies (depending on the case).

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)}$ DIVERGES – integral or limit comparison test (compare to $\sum_{n=1}^{\infty} \frac{1}{n}$).

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{(n^2+1)}$ CONVERGES – alternating series test

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ CONVERGES – integral, comparison (e.g. $\sum_{n=1}^{\infty} \frac{1}{n^3}$), or limit comparison test

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(n+\frac{1}{n}\right)^n$ DIVERGES – nth term test ($\lim_{n\to\infty} \left(n+\frac{1}{n}\right)^n \neq 0$)

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{5n^2}$ DIVERGES – limit comparison test, nth term test, or comparison test (with $\sum_{n=1}^{\infty} \frac{1}{5}$)

   (f) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ (hint: consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$) CONVERGES – use the fact that $|\sin(x)| \leq x$ for $x \geq 0$. Then show the series is absolutely convergent using the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ CONVERGES – ratio test or notice that this is the taylor series for e^x evaluated at $x=2$.

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n+3)!}$ DIVERGES – ratio test, or, nth term test: $2n \geq n+3$ whenever $n \geq 3$. So $(2n)! \geq (n+3)!$ whenever $n \geq 3$, and $\lim_{n\to\infty} \frac{(2n)!}{(n+3)!} \neq 0$.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$ CONVERGES – limit comparison test or comparison test ($\frac{n!}{(n+2)!} = \frac{1}{(n+2)(n+1)} \leq \frac{1}{n^2}$. Then use the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.)

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ CONVERGES – ratio test.

$$\lim_{n\to\infty} \frac{\left|\frac{(n+1)!}{(n+1)^{n+1}}\right|}{\left|\frac{n!}{n^n}\right|} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right)^{-1} = \frac{1}{e} < 1.$$

2. Find the values of a for which the series converges/diverges:

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^a}$ Converges if and only if $a > 1$. Use the integral test. For $a \neq 1$, use $\int \frac{1}{x}(\ln|x|)^{-a}dx = \frac{1}{-a+1}(\ln|x|)^{-a+1} + C$. For $a = 1$, consider $\ln(\ln x)$.

(b) $\sum_{n=1}^{\infty} \frac{1}{(n!)^a}$ Converges if and only if $a > 0$. Use the ratio test for the case $a \neq 0$ (the case $a = 0$ is easier).

3. Consider the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Are the following statements true or false? Fully justify your answer.

   The series does not converge (use the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$).

   (a) The series converges by limit comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$. False

   (b) The series converges by the ratio test. False

   (c) The series converges by the integral test. False

4. Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$. Are the following statements true or false? Fully justify your answer.

   (a) The series converges by limit comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$. False

   (b) The series converges by the ratio test. False

   (c) The series converges by the integral test. False

   (d) The series converges by the alternating series test. True

5. The series $\sum a_n$ is absolutely convergent. Are the following true or false? Explain.

   (a) $\sum a_n$ is convergent. True

   (b) The sequence a_n is convergent. True

   (c) $\sum (-1)^n a_n$ is convergent. True

   (d) The sequence a_n converges to 1. False

   (e) $\sum a_n$ is conditionally convergent. False

   (f) $\sum \frac{a_n}{n}$ converges. True

6. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

   You must justify your answer to receive credit. CONVERGES – use the ratio test: if a_n denotes the nth term of this series, then $\lim_{n\to\infty} |a_{n+1}/a_n| = 0$.

7. Let $f(x) = \frac{1}{1-x}$.

(a) Find an upper bound M for $\dfrac{|f^{(n+1)}(x)|}{(n+1)!}$ on the interval $(-1/2, 1/2)$. 2^{n+2}. With $f(x) = 1/(1-x)$, have $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$. We can see the function $|f^{(n)}(x)|$ is an increasing function on $[-1/2, 1/2]$ so that

$$\frac{|f^{(n+1)}(x)|}{(n+1)!} \leq \frac{(n+1)!}{(1-\frac{1}{2})^{n+2}(n+1)!} = 2^{n+2}.$$

(b) Use this result to show that the Taylor series for $\dfrac{1}{1-x}$ converges to $\dfrac{1}{1-x}$ on the interval $(-1/2, 1/2)$.

Using the previous problem, we have for any $-1/2 < x < 1/2$,

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \right| \leq \frac{\left((N+1)! \cdot 2^{N+2}\right) |x|^{N+1}}{(N+1)!} = 2^{N+2}|x|^{N+1} = 2(2|x|)^{N+1}.$$

Observe that for $-1/2 < x < 1/2$,

$$\lim_{N\to\infty} 2(2|x|)^{N+1} = 0.$$

Thus for any $-1/2 < x < 1/2$, we can conclude that

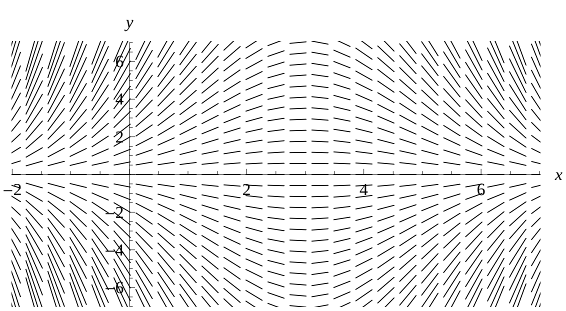
$$\lim_{N\to\infty} \left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \right| = 0.$$

8. If $\sum b_n(x-2)^n$ converges at $x = 0$ but diverges at $x = 7$, what is the largest possible interval of convergence of this series? What's the smallest possible? Largest: $[-3, 7)$. Smallest: $[0, 4)$.

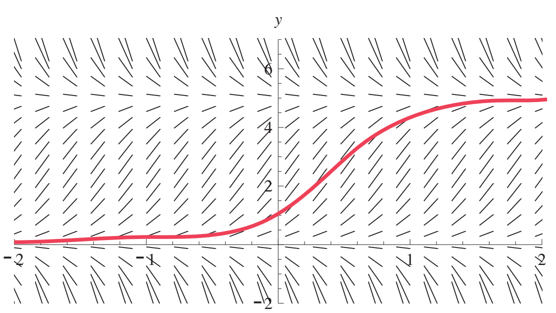
9. (a) Which of the slope fields (i)–(iv) below could be the slope field for the "logistic" differential equation

$$\frac{dy}{dx} = y(5-y) \text{ ?}$$

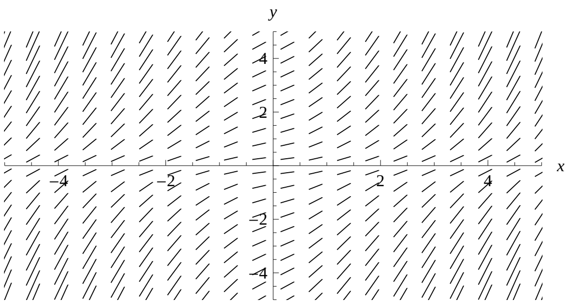
Please explain how you got your answer.



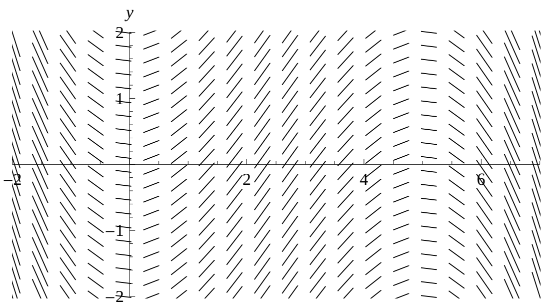
(i)



(ii)



(iii)



(iv)

Slope field (ii): this is the only one that has slope zero along the lines $y = 0$ and $y = 5$.

(b) On the slope field you chose for part (a) of this problem, sketch in the solution curve for the above logistic differential equation that has initial condition $y(0) = 1$.

10. (a) Write down the second degree Taylor polynomial $P_2(x)$ approximating

$$f(x) = \ln\big(1 + x(1 - x)\big)$$

near $x = 0$. $P_2(x) = x - \dfrac{3x^2}{2}$. To see this, we can use the fact that

$$\ln(1+z) + C = \int \frac{1}{1+z}dz = \int (1 - z + z^2 - \ldots)dz = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots + D$$

Setting $z = 0$, we see that

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots.$$

Taking $z = x(1-x)$ we have

$$\ln(1 + x(1-x)) = x(1-x) - \frac{x^2(1-x)^2}{2} + \ldots = x - \frac{3}{2}x^2 + \ldots$$

In other words, the second degree Taylor polynomial is

$$P_2(x) = x - \frac{3}{2}x^2.$$

Alternatively, with $f(x) = \ln(1 - x(1-x))$, one can compute directly that $f(0) = 0$, $f'(0) = 1$, and $f''(0) = -3$.

(b) Use your result from part (a) to approximate $\ln(1.09)$. Hint: $\frac{1}{10} \cdot \frac{9}{10} = 0.09$. $\ln(1.09) = f(0.1) \approx P_2(0.1) = 0.085$. To see this: From the definition of $f(x)$, we have $\ln(1.09) = f(1/10)$. Approximating $f(x)$ with $P_2(x)$, we have

$$f(1/10) \approx P_2(1/10) = \frac{1}{10} - \frac{3}{2} \cdot \left(\frac{1}{10}\right)^2 = \frac{17}{200} = \frac{1}{2} \cdot 0.17 = 0.085.$$

(c) What does Lagrange's error bound say about the error in the approximation you found in part (b)? You should find it useful to note that

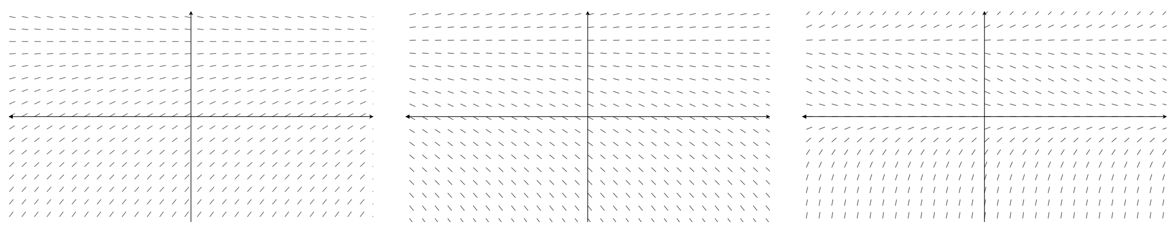
$$f'''(x) = \frac{2(2x-1)\left(x^2 - x + 4\right)}{\left(x^2 - x - 1\right)^3},$$

and that $f'''(x)$ is a *decreasing* function on the interval $(0, 1/10)$. Using the hint, we can bound $|f'''(x)|$ on $[0, 1/10]$ by $f'''(0) = 8$ (note that $f'''(1/10) > 0$). Thus we have

$$|f(x) - P_2(x)| \leq \frac{8 \cdot x^3}{3!} \leq \frac{4}{3}\frac{1}{1000} = 0.001\bar{3}$$

on $[0, 1/10]$. Thus we may conclude that $|\ln(1.09) - 0.085| \leq 0.001\bar{3}$.

11. Show below are the slope fields of three differential equations, "A", "B", and "C". For each slope field, the axes intersect at the origin.



A B C

For each of the following functions, indicate which, if any, of the differential equations, "A", "B", and "C" it could be the solution of. Note that any of the functions could be a solution to zero, one, or more than one of the differential equations. If a function is a solution to none of the differential equations clearly write "None" as your anwer.

(a) $y = 0$: A,B
(b) $y = 1$: B,C
(c) $y = 1 + ke^x$: None (unless $k = 0$)

For the pictures above, which are different from the ones on the review:

(a) $y = 0$: C
(b) $y = 1$: A,B,C
(c) $y = 1 + ke^x$: B, or C with $k \geq 0$

12. Solve (make sure to write your final answer in the form $y =$ a function of x):

We include general solutions to help check your work. **The solution to each problem is the particular solution**.

(a) $\dfrac{dy}{dx} = -\dfrac{x}{y^2}$, $y(2) = 1$ General solution $y = \sqrt[3]{-\frac{3}{2}x^2 + C}$. Particular solution $y = \sqrt[3]{-\frac{3}{2}x^2 + 7}$

(b) $y \cdot y' = x(1 + y^2)$, $y(1) = 2$ General solution $y = \sqrt{Ce^{x^2} - 1}$. Particular solution $y = \sqrt{\frac{5}{e}e^{x^2} - 1}$.

(c) $y' = y\cos(x)$, $y(0) = 3$ General solution $y = Ce^{\sin x}$. Particular solution $y = 3e^{\sin x}$.

(d) $(x^3 + 1)\dfrac{dy}{dx} - 3x^2 = 0$, $y(1) = \ln 2$ General solution $y = \ln(x^3 + 1) + C$. Particular solution $y = \ln(x^3 + 1)$. To see this, we start with

$$(x^3 + 1)\frac{dy}{dx} - 3x^2 = 0.$$

We then get

$$dy = \frac{3x^2}{x^3+1}dx.$$

Integrating we have

$$y = \ln(x^3+1) + C$$

gives a solution to the problem (for $x > -1$). For the particular solution, we want

$$y(1) = \ln(1+1) + C = \ln(2).$$

It follows that $C = 0$.

(e) $\frac{1}{e^{y^3+1}}\frac{dy}{dx} - \frac{1}{3y^2} = 0$, $y(1) = e^2$ General solution $y = \sqrt[3]{-1-\ln(-x+C)} = -\sqrt[3]{\ln(-ex+D)}$.
Particular solution $y = -\sqrt[3]{\ln(-ex + e^{-e^6} + e)}$

13. Consider a continuous function $f(x)$ with $f(0) = 1$ and $f(1) = 2$. Consider the solution of the differential equation $f(x)\frac{dy}{dx} - f'(x) = 0$, which satisfies the initial condition $y(0) = 1$. What is the value of this solution at $x = 1$? Then answer is $\ln(2) + 1$. From the differential equation, we have

$$y = \ln(f(x)) + C.$$

Then

$$y(0) = \ln(f(0)) + C = \ln(1) + C = C = 1.$$

It follows that

$$y = \ln(f(x)) + 1.$$

Finally, we have

$$y(1) = \ln(f(1)) + 1 = \ln(2) + 1.$$

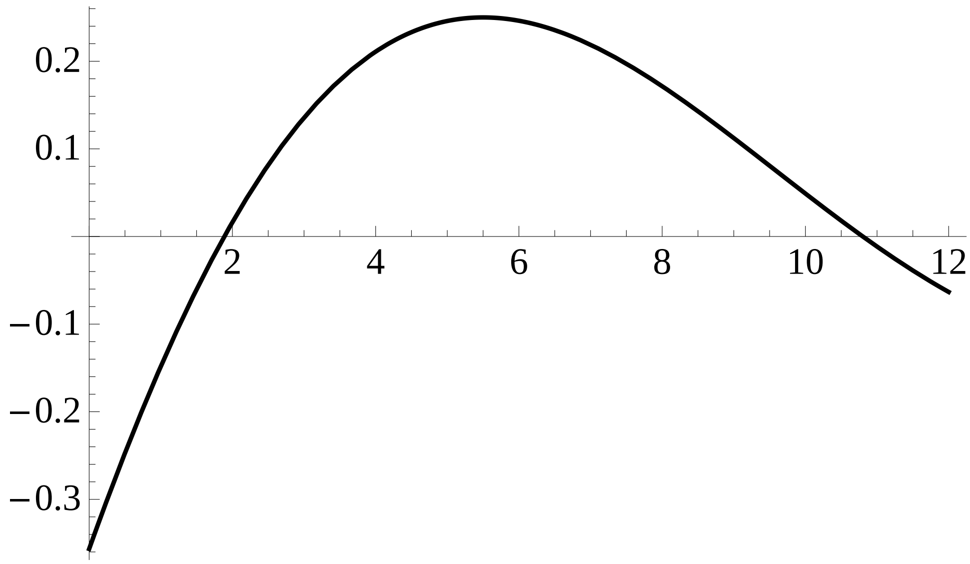
14. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{(x+4)^n}{n^2}$$

Find the intervals of convergence of f and f'.

For f: $[-5, -3]$. For f': $[-5, -3)$. Using the ratio test for f, you can check that the radius of convergence is 1. When $x = -5$ the resulting series converges using the alternating series test. When $x = -3$, the series converges using the integral test. A similar analysis gives the result for f'; when $x = -5$, the series converges using the alternating series test. When $x = -3$, the series diverges using the integral test.

15. Consider the function $y = f(x)$ sketched below.



Suppose $f(x)$ has Taylor series

$$f(x) = a_0 + a_1(x-4) + a_2(x-4)^2 + a_3(x-4)^3 + \ldots$$

about $x = 4$.

(a) Is a_0 positive or negative? Please explain. $a_0 > 0$, because the function is positive at $x = 4$.

(b) Is a_1 positive or negative? Please explain. $a_1 > 0$, because the function is increasing at $x = 4$.

(c) Is a_2 positive or negative? Please explain. $a_2 < 0$, because the function is concave down at $x = 4$.

16. How many terms of the Taylor series for $\ln(1+x)$ centered at $x = 0$ do you need to estimate the value of $\ln(1.4)$ to three decimal places?

8 terms (the degree 7 Taylor polynomial) will suffice (to make the error at most 10^{-4}). We have seen above that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}.$$

We may use the error bound for Taylor series, but since this is an alternating series, it is faster to use the alternating series test (and you can check you will get the same bound). From this we obtain

$$\left| \ln(1 + 4/10) - \sum_{n=1}^{N} (-1)^n \frac{\left(\frac{4}{10}\right)^n}{n} \right| \leq \frac{\left(\frac{4}{10}\right)^{N+1}}{N+1},$$

which we want to be at most 10^{-4}. That is we want to find N such that

$$\frac{\left(\frac{4}{10}\right)^{N+1}}{N+1} \leq \frac{1}{10^4}.$$

One can check this is the case for $N \geq 7$.

17. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second degree Taylor polynomial, estimate how far the car moves in the next second. The car moves about 21 m. The second degree Taylor approximation of the position at time t is given by $s(t) = s(0) + 20t + \frac{2}{2}t^2$, where $s(0)$ is the initial position.

18. Find the integral and express the answer as an infinite series.

$$\int \frac{e^x - 1}{x}\, dx = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C$$

19. Using series, evaluate the limit

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots) - x}{x^3} = \lim_{x\to 0} \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^3}$$
$$= \lim_{x\to 0} \frac{-\frac{1}{3!} + \frac{x^2}{5!} - \dots}{1} = -\frac{1}{6}.$$

20. Use the Lagrange Error Bound for $P_n(x)$ to find a reasonable bound for the error in approximating the quantity $e^{0.60}$ with a third degree Taylor polynomial for e^x centered at $x = 0$.

For $n = 3$, the Lagrange error bound is given by $\frac{Mx^4}{4!}$, where M is the maximum of $|f^4(u)| = |e^u|$ on the interval between 0 and x. For $x = 0.6$, M is the maximum of e^y (an increasing function) on the interval $[0, 0.6]$, so $M = e^{0.6}$. The bound is: $\frac{e^{0.6}(0.6)^4}{4!}$.

21. Consider the error in using the approximation $\sin\theta \approx \theta - \theta^3/3!$ on the interval $[-1, 1]$. Where is the approximation an overestimate? Where is it an underestimate?

For $0 \le \theta \le 1$, the estimate is an underestimate (the alternating Taylor series for $\sin\theta$ is truncated after a negative term). For $-1 \le \theta \le 0$, the estimate is an overestimate (the alternating Taylor series is truncated after a positive term).

22. Find the Taylor series around $x = 0$ for

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

(Your answer should involve only even powers of x.)

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

One obtains this in the following way:

$$\frac{\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} (1 + (-1)^n)\frac{x^n}{n!}\right) = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}$$