MATH 2300 – review problems for Exam 3

1. Check whether the following series converge or diverge. In each case, justify your answer by either computing the sum or by by showing which convergence test you are using, why and how it applies (depending on the case).

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{(n^2+1)}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(n+\frac{1}{n}\right)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{5n^2}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ (hint: consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$)

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n+3)!}$

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$

   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

2. Find the values of a for which the series converges/diverges:

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^a}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n!)^a}$

3. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Are the following statements true or false? Fully justify your answer.

   (a) The series converges by limit comparison with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

   (b) The series converges by the ratio test.

(c) The series converges by the integral test.

4. Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$. Are the following statements true or false? Fully justify your answer.

   (a) The series converges by limit comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.
   (b) The series converges by the ratio test.
   (c) The series converges by the integral test.
   (d) The series converges by the alternating series test.

5. The series $\sum a_n$ is absolutely convergent. Are the following true or false? Explain.

   (a) $\sum a_n$ is convergent.
   (b) The sequence a_n is convergent.
   (c) $\sum (-1)^n a_n$ is convergent.
   (d) The sequence a_n converges to 1.
   (e) $\sum a_n$ is conditionally convergent.
   (f) $\sum \frac{a_n}{n}$ converges.

6. Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

   You must justify your answer to receive credit.

7. Let $f(x) = \frac{1}{1-x}$.

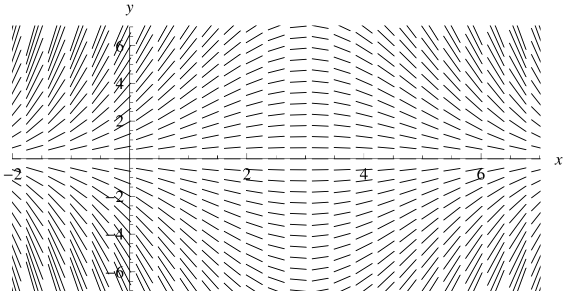
   (a) Find an upper bound M for $\frac{|f^{(n+1)}(x)|}{(n+1)!}$ on the interval $(-1/2, 1/2)$.
   (b) Use this result to show that the Taylor series for $\frac{1}{1-x}$ converges to $\frac{1}{1-x}$ on the interval $(-1/2, 1/2)$.

8. If $\sum b_n (x-2)^n$ converges at $x = 0$ but diverges at $x = 7$, what is the largest possible interval of convergence of this series? What's the smallest possible?

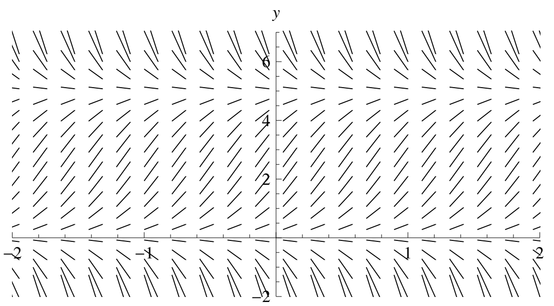
9. (a) Which of the slope fields (i)–(iv) below could be the slope field for the "logistic" differential equation

$$\frac{dy}{dx} = y(5-y) \ ?$$

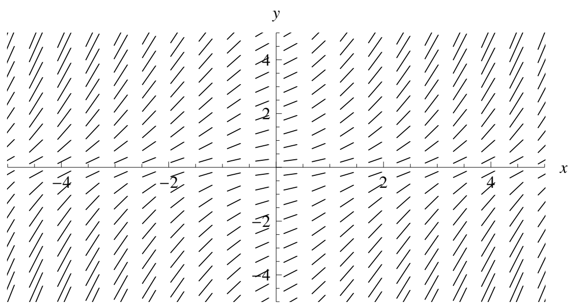
   Please explain how you got your answer.



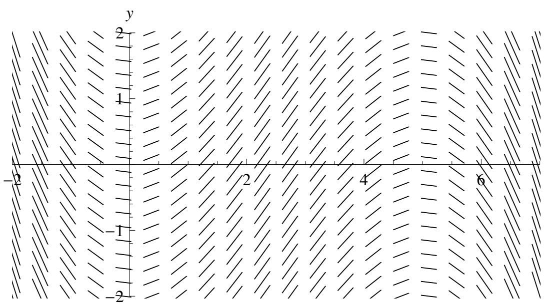
(i)



(ii)



(iii)



(iv)

(b) On the slope field you chose for part (a) of this problem, sketch in the solution curve for the above logistic differential equation that has initial condition $y(0) = 1$.

10. (a) Write down the second degree Taylor polynomial $P_2(x)$ approximating

$$f(x) = \ln\big(1 + x(1 - x)\big)$$

near $x = 0$.

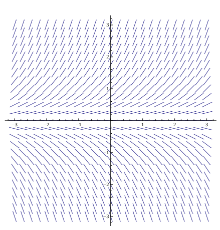
(b) Use your result from part (a) to approximate $\ln(1.09)$. Hint: $\frac{1}{10} \cdot \frac{9}{10} = 0.09$.

(c) What does Lagrange's error bound say about the error in the approximation you found in part (b)? You should find it useful to note that

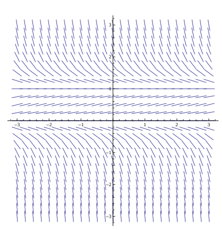
$$f'''(x) = \frac{2(2x-1)\left(x^2 - x + 4\right)}{\left(x^2 - x - 1\right)^3},$$

and that $f'''(x)$ is a *decreasing* function on the interval $(0, 1/10)$.

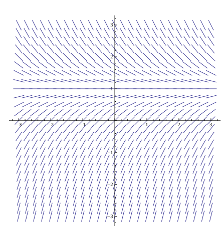
11. Show below are the slope fields of three differential equations, "A", "B", and "C". For each slope field, the axes intersect at the origin.



A



B



C

For each of the following functions, indicate which, if any, of the differential equations, “A”, “B”, and “C” it could be the solution of. Note that any of the functions could be a solution to zero, one, or more than one of the differential equations. If a function is a solution to none of the differential equations clearly write ”None” as your answer.

(a) $y = 0$

(b) $y = 1$

(c) $y = 1 + ke^x$

12. Solve (make sure to write your final answer in the form y = a function of x):

(a) $\dfrac{dy}{dx} = -\dfrac{x}{y^2}$, $y(2) = 1$

(b) $y \cdot y' = x(1 + y^2)$, $y(1) = 2$

(c) $y' = y\cos(x)$, $y(0) = 3$

(d) $(x^3 + 1)\dfrac{dy}{dx} - 3x^2 = 0$, $y(1) = \ln 2$

(e) $\dfrac{1}{e^{y^3+1}}\dfrac{dy}{dx} - \dfrac{1}{3y^2} = 0$, $y(1) = e^2$

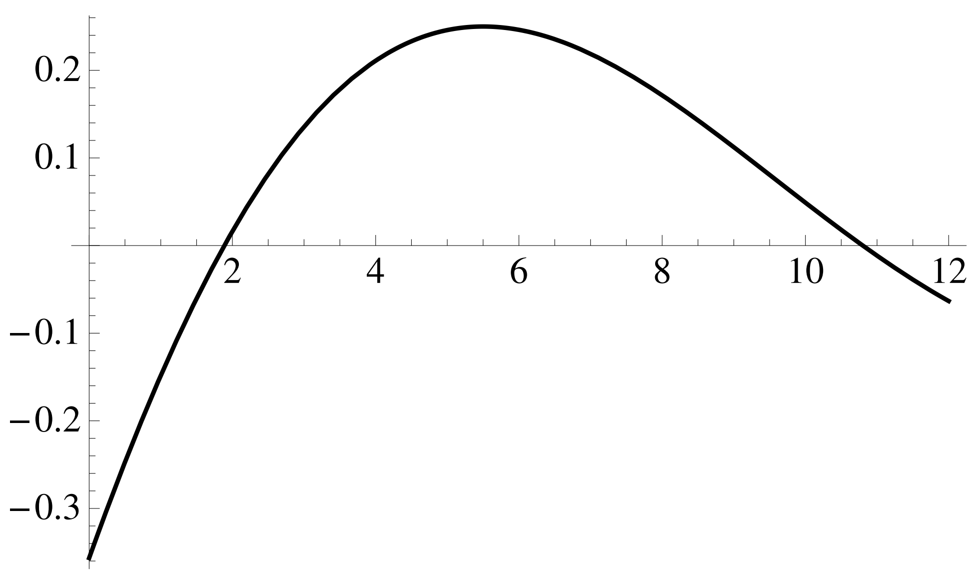
13. Consider a continuous function $f(x)$ with $f(0) = 1$ and $f(1) = 2$. Consider the solution of the differential equation $f(x)\dfrac{dy}{dx} - f'(x) = 0$, which satisfies the initial condition $y(0) = 1$. What is the value of this solution at $x = 1$?

14. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{(x+4)^n}{n^2}$$

Find the intervals of convergence of f and f'.

15. Consider the function $y = f(x)$ sketched below.



Suppose $f(x)$ has Taylor series

$$f(x) = a_0 + a_1(x-4) + a_2(x-4)^2 + a_3(x-4)^3 + ...$$

about $x = 4$.

(a) Is a_0 positive or negative? Please explain.

(b) Is a_1 positive or negative? Please explain.

(c) Is a_2 positive or negative? Please explain.

16. How many terms of the Taylor series for $\ln(1+x)$ centered at $x = 0$ do you need to estimate the value of $\ln(1.4)$ to three decimal places?

17. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second degree Taylor polynomial, estimate how far the car moves in the next second.

18. Find the integral and express the answer as an infinite series.

$$\int \frac{e^x - 1}{x}\, dx$$

19. Using series, evaluate the limit

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

20. Use the Lagrange Error Bound for $P_n(x)$ to find a reasonable bound for the error in approximating the quantity $e^{0.60}$ with a third degree Taylor polynomial for e^x centered at $x = 0$.

21. Consider the error in using the approximation $\sin\theta \approx \theta - \theta^3/3!$ on the interval $[-1, 1]$. Where is the approximation an overestimate? Where is it an underestimate?

22. Find the Taylor series around $x = 0$ for

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

(Your answer should involve only even powers of x.)