MATH 2300 – review problems for Exam 2

1. A pyramid of constant density δ gm/cm^3 has a square base of sidelength 40 cm, and a height of 10 cm.
   (a) Find the mass of the pyramid.
   (b) Find the center of mass of the pyramid.

2. A metal plate of constant density 5 gm/cm^2 has a shape bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 1$.
   (a) Find the mass of the plate.
   (b) Find the center of mass of the plate.

3. Find the mass of the plate in the previous problem if, instead of constant density, the plate has density:
   (a) $\delta(x) = 1 + x$,
   (b) $\delta(y) = 1 + y$.

4. Find the area of the region bounded by the curve $r = \sqrt{\theta}$, for $0 \leq \theta \leq \pi$.

5. Find the area of the region that lies within the limaçon $r = 3 + 2\cos(\theta)$ and outside the circle $r = 4$.

6. Find the area of the region common to the circles $r = \cos(\theta)$ and $r = \sqrt{3}\sin(\theta)$.
   (Hint: $\tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$.)

7. Find the exact length of the polar curve $r = e^{\theta}$, for $0 \leq \theta \leq \pi/2$.

8. Let $\{f_n\}$ be the sequence defined recursively by $f_1 = 5$ and $f_n = f_{n-1} + 2n + 4$.
   (a) Check that the sequence g_n whose n-th term is $g_n = n^2 + 5n - 1$ satisfies this recurrence relation, and that $g_1 = 5$. (This tells us $g_n = f_n$ for all n.)
   (b) Use the result of part (a) to find f_{20} quickly, with the aid of a calculator.

9. Decide whether each of the following sequences converges. If a series converges, what does it converge to? If not, why not?
   (a) The sequence whose n-th term is $a_n = 1 - \frac{1}{n}$.
   (b) The sequence whose n-th term is $b_n = \sqrt{n+1} - \sqrt{n}$.
   (c) The sequence whose n-th term is $c_n = \cos(\pi n)$.
   (d) The sequence $\{d_n\}$, where $d_1 = 2$ and
   $$d_n = 2d_{n-1} \quad \text{for } n > 1.$$

10. Consider the region defined in polar coordinates by $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq \dfrac{1}{\sin(\theta) + \cos(\theta)}$.
    (a) Sketch a graph of this region.
    (b) Use your graph to find the area. (The integral in polar coordinates is hard!)

11. Find the arc length of the part of the cardioid $r = 1 + \cos(\theta)$ where $0 \le \theta \le \pi/2$.

12. The following integrals represent the area of some region in the xy plane. Draw a graph of the area, labeling the axes and giving the equation(s) of the function(s).

    (a) $\int_{-2}^{0} (-4x)\, dx$

    (b) $\int_{-3}^{3} (-\sqrt{9 - x^2})\, dx$

    (c) $\int_{1}^{2} 3y\, dy$

    (d) $\int_{0}^{1} (\sqrt{y} - y)\, dy$

13. Using slices parallel to the base, write a definite integral representing the volume of a cone with a height of 10 cm and a base of diameter 6 cm.

14. A ball is dropped from a height of 10 feet and bounces. Assume that there is no air resistance. Each bounce is $\frac{3}{4}$ of the height of the bounce before.

    (1) Find an expression for the height to which the ball rises after it hits the floor for the nth time. (2) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the nth time. (3) Using without proof the fact that a ball dropped from a height of h feet reaches the ground in $\sqrt{h}/4$ seconds: Will the ball bounce forever? If not, how long it will take for the ball to come to rest?

15. In theory, drugs that decay exponentially always leave a residue in the body. However in practice, once the drug has been in the body for 5 half-lives, it is regarded as being eliminated. If a patient takes a tablet of the same drug every 5 half-lives forever, what is the upper limit to the amount of drug that can be in the body?

16. Derive the volume formulas for the following shapes by using an appropriate integral:

    (a) a circular cylinder of height H whose radius is R,

    (b) a circular cone of height H whose radius at the base is R,

    (c) a square pyramid of height H whose base has side length S,

    (d) a triangular pyramid of height H whose base is an equilateral triangle with side length S,

    (e) a sphere of radius R.

17. Suppose that $0 \le f(x) \le g(x)$ for $x \ge a$. If $\int_a^\infty f(x)\mathrm{d}x$ converges and $\int_a^\infty g(x)\mathrm{d}x$ diverges, then is the area between the curves $f(x)$ and $g(x)$ for $x \ge a$ finite or infinite?

18. Determine whether the following integrals converge or diverge:

    (a) $\int_1^\infty \frac{5 - 2\sin(e^x)}{x^2}\mathrm{d}x$

    (b) $\int_1^\infty \frac{1}{x + \ln x}\mathrm{d}x$

(c) $\displaystyle\int_1^{\infty} \frac{x^2+x+1}{x^5+3x^2+1}\mathrm{d}x$

(d) $\displaystyle\int_1^{\infty} \frac{1}{e^x-x}\mathrm{d}x$

(e) $\displaystyle\int_0^{\infty} \frac{e^x}{e^{2x}+1}\mathrm{d}x$

19. Consider the region bounded by $y=\sqrt{x}$, $y=0$, $x=1$.

    (a) Sketch the solid obtained by rotating the above region around the x-axis.

    (b) Using the sketch, write a Riemann sum approximating the volume of the solid.

    (c) Convert your sum into an integral and find the volume.

    (d) Repeat parts (a)-(c) with the same region rotated around the y-axis.

20. Using the table below, estimate the length of the curve given by $y=f(x)$ from $(3,4)$ to $(6,0.7)$.

x	3	3.5	4	4.5	5	5.5	6
$f(x)$	4	3.6	2.4	-1	-0.5	0	0.7
$f'(x)$	-0.8	-2.4	-6.8	1	1	1.4	-0.4

21. Find the volume of the solid whose base is the region in the xy-plane bounded by the curves $y=x$ and $y=x^2$ and whose cross sections perpendicular to the x-axis are squares with one side in the xy-plane.

22. Do the same thing as the previous problem except with semi-circle cross sections and then again with cross sections that are isosceles triangles of height 3.

23. A steady wind blows a kite due east. The kite's height above ground from horizontal position $x=0$ to $x=80$ feet is given by

$$y = 150 - \frac{1}{40}(x-50)^2$$

    Find the distance traveled by the kite.

24. Supplemental Problems from Section 8.2: 17, 21

25. A certain bacteria is growing in a petri dish of volume 300 cm^3. Assume that each bacterium occupies 1×10^{-12} cm^3, and that the growth rate of the bacteria, starting at time $t=1$ hour, is given by

$$r(t) = \frac{\sin(t)+2}{t}\ \frac{\text{bacteria}}{\text{hour}}.$$

    Will the bacteria ever outgrow the petri dish? Explain your answer carefully.

26. Show that the volume contained in the solid obtained by rotating the curve $y=e^{-x}$, from $x=1$ to ∞, about the x-axis is finite.

27. The density of oil in a circular oil slick on the surface of the ocean at a distance r meters from the center of the slick is given by $\delta(r) = 50/(1+r)$ kg/m^2.

(a) If the slick extends from $r = 0$ to $r = 10,000$ m, find a Riemann sum approximating the total mass of the oil in the slick.

(b) Find the exact value of the mass of oil in the slick by turning your sum into an integral, and evaluating it.

(c) Within what distance r is half of the oil slick contained?

28. Determine if the following statements are true or false. If the statement is true, clearly explain why. If the statement is false, give an example exhibiting why it is false.

    (a) If f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)\,dx$ converges.

    (b) If f is continuous on $[0, \infty)$ and $\int_0^\infty f(x)\,dx$ diverges, then $\lim_{x\to\infty} f(x) \neq 0$.

    (c) If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f'(x)\,dx = -f(0)$.

    (d) If $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ both converge, then $\int_0^\infty (f(x) + g(x))\,dx$ converges.

    (e) If $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ both diverge, then $\int_0^\infty (f(x) + g(x))\,dx$ diverges.

29. Compute the Taylor polynomial $P_5(x)$ for each of the following functions:

    (a) $\sin(x^2)$ at $x = 0$,

    (b) $x^{1/3}$ at $x = 1$,

    (c) $\ln(\cos(x))$ at $x = \pi$.

30. (a) Write down the second degree Taylor polynomial $P_2(x)$ approximating

$$f(x) = \ln\big(1 + x(1 - x)\big)$$

    near $x = 0$.

    (b) Use your result from part (a) to approximate $\ln(1.09)$. Hint: $\frac{1}{10} \cdot \frac{9}{10} = 0.09$.