MATH 2300 – review problems for Exam 1

1. Evaluate the integral $\int \sin x \cos x \, dx$ in each of the following ways:

   (a) Integrate by parts, with $u = \sin x$ and $dv = \cos x \, dx$. The integral you get on the right should look much like the one you started with, so you can solve for this integral.

   (b) Integrate by parts, with $u = \cos x$ and $dv = \sin x \, dx$.

   (c) Substitute $w = \sin x$.

   (d) Substitute $w = \cos x$.

   (e) First use the fact that $\sin x \cos x = \frac{1}{2}\sin(2x)$, and then antidifferentiate directly.

   (f) Show that answers to parts (a)–(e) of this problem are all the same. It may help to use the identities $\cos^2 x + \sin^2 x = 1$ and $\cos(2x) = 1 - 2\sin^2 x$.

2. Let $f(x)$ be a continuous function on the set of all real numbers. Show that

$$\int_0^1 f(e^x)\, e^x \, dx = \int_1^e f(x)\, dx.$$

3. (a) Explain why the integral

$$\int_2^5 \frac{x \, dx}{\sqrt{x^2 - 4}}$$

   is improper.

   (b) Show that

$$\int_2^5 \frac{x \, dx}{\sqrt{x^2 - 4}} = \sqrt{21}.$$

4. Suppose that $\int_0^1 f(t)dt = 5$. Calculate the following:

   (a) $\int_0^{0.5} f(2t)dt$

   (b) $\int_0^1 f(1-t)dt$

   (c) $\int_1^{1.5}(3-2t)dt$

5. Evaluate the following integrals:

   (a) $\displaystyle\int 2x \cos(x^2)\, dx$

   (b) $\displaystyle\int e^{2x} \sin(2x)\, dx$

   (c) $\displaystyle\int \cos^2 \theta \, d\theta$

   (d) $\displaystyle\int x^2 \sin(x) dx$

   (e) $\displaystyle\int \frac{1}{\sqrt{x^2 - 16}}\, dx$

   (f) $\displaystyle\int \frac{x}{\sqrt{6x - x^2}}\, dx$

(g) $\displaystyle\int \frac{3x^2+6}{x^2(x^2+3)}\,dx$

(h) $\displaystyle\int \sqrt{25-x^2}\,dx$

(i) $\displaystyle\int \frac{3x-1}{x^2-5x+6}\,dx$

(j) $\displaystyle\int \sin^3(5x)\cos(5x)\,dx$

(k) $\displaystyle\int_2^3 \frac{x^2}{1+x^3}\,dx$

(l) $\displaystyle\int_0^3 x\,e^{x^2}\,dx$

(m) $\displaystyle\int x^7 e^{x^4}\,dx$

(n) $\displaystyle\int (\ln(x))^2\,dx$

6. Evaluate the following integrals, using the substitutions provided.

   (a) $\int y\sqrt{y^2+1}\,dy$; $w = y^2+1$.

   (b) $\int y\sqrt{y+1}\,dy$; $w = y+1$.

7. (a) Calculate $\displaystyle\int_2^4 \frac{dx}{(x-3)^2}$, if it exists.

   (b) Find $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\,dx$, if it converges.

8. Estimate $\displaystyle\int_1^2 \ln x\,dx$, by subdividing the interval $[1,2]$ into eight equal parts, and using:

   (a) A left-hand Riemann sum:

   (b) A right-hand Riemann sum:

   (c) The midpoint rule:

   (d) The trapezoid rule:

9. Estimate $\displaystyle\int_0^2 e^{x^2}\,dx$ by subdividing the interval $[0,2]$ into four equal parts, and using:

   (a) A left-hand Riemann sum:

   (b) A right-hand Riemann sum:

   (c) The midpoint rule:

   (d) The trapezoid rule:

10. Using the table, estimate the total distance traveled from time $t = 0$ to time $t = 6$ using the trapezoidal rule and the midpoint rule. Divide the interval $[0,6]$ into three equal parts.

Time, t	0	1	2	3	4	5	6
Velocity, v	4	5	6	8	9	5	3

11. Consider the function $f(x) = x^2 + 3$ on the interval $[0, 1]$. Determine whether each of the following four methods of integral approximation will give an overestimate or underestimate of $\int_0^1 f(x)dx$. In each case, draw a picture to justify your answer.

    (a) the left Riemann sum

    (b) the right Riemann sum

    (c) the trapezoidal rule

    (d) the midpoint rule

12. Suppose $f(x)$ is concave up and decreasing on the interval $[0, 1]$. Suppose the approximations LEFT(100), RIGHT(100), MID(100), and TRAP(100) yield the following estimates for $\int_0^1 f(x)\,dx$: 1.10, 1.25, 1.35, and 1.50, but *not necessarily in that order*. Which estimate do you think came from which method? Please explain your reasoning.

13. Which of the following integrals can be integrated using partial fractions?

    (a) $\int \frac{1}{x^4-3x^2+2}\,dx$

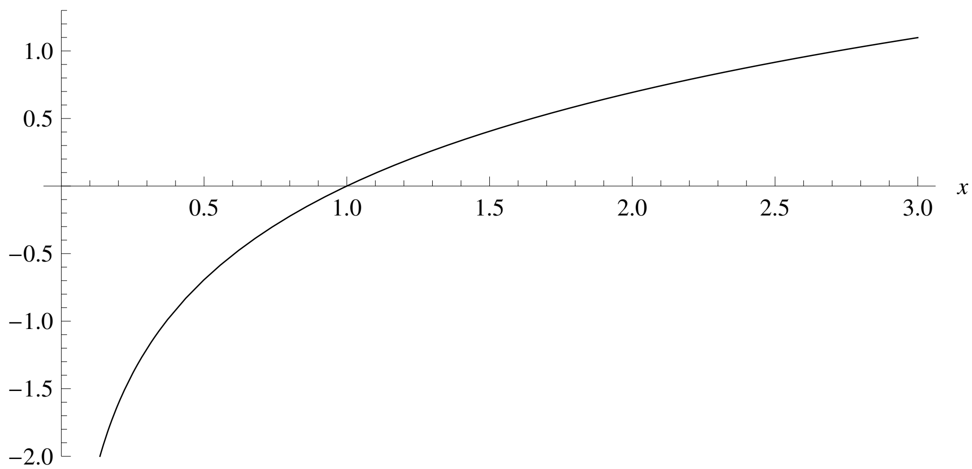
    (b) $\int \frac{1}{x^4+1}\,dx$

    (c) $\int \frac{1}{x^3-8}\,dx$

    (d) $\int \frac{1}{x^4+2x^2+2}\,dx$

    Make sure you can show the partial fraction decomposition.

14. Suppose $f(x)$ is a function whose graph looks like this:



Suppose the approximations LEFT(100), RIGHT(100), MID(100), and TRAP(100) yield the following estimates for $\int_1^2 f(x)\,dx$: 0.3423, 0.3857, 0.3866, 0.4920, but *not necessarily in that order.* Which estimate do you think came from which method? Between which two estimates do you think the true value of $\int_1^2 f(x)\,dx$ lies? Please explain your reasoning.

15. For this set of problems, state which techniques are useful in evaluating the integral. You may choose from: integration by parts; partial fractions; long division; completing the square; trig substitution; or another substitution. There may be multiple answers.

(a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int \frac{1}{\sqrt{6x-x^2-8}}\,dx$

(c) $\displaystyle\int x\sin x\,dx$

(d) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

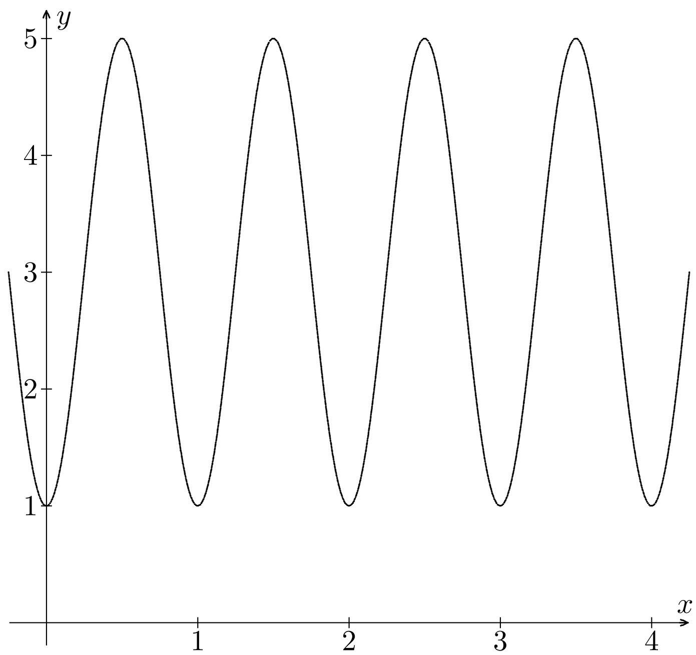
(e) $\displaystyle\int \frac{x^2}{1-x^2}\,dx$

(f) $\displaystyle\int (1+x^2)^{-3/2}\,dx$

(g) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

(h) $\displaystyle\int \frac{1}{1-x^2}\,dx$

16. This problem relies on the following graph:



Using two subintervals, estimate $\int_0^4 f(x)\,dx$ with the left, right, midpoint, and trapezoid approximations. (If the picture was drawn correctly, you should find that they're all equal.)

(a) Both the left and right approximations with two rectangles underestimate the actual value integral. How is this possible (i.e., why isn't one of them an overestimate)?

(b) Both the midpoint and trapezoid approximations with two rectangles underestimate the actual value of the integral. How is this possible (i.e., why isn't one of them an overestimate)?

17. A patient is given an injection of Imitrex, a migraine medicine, at a rate of $r(t) = 2te^{-2t}$ ml/sec, where t is the number of seconds since the injection started.

(a) By letting $t \to \infty$, estimate the total quantity of Imitrex injected.

(b) What fraction of this dose has the patient received at the end of 5 seconds?

18. Let f be a differentiable function. Suppose that $f''(0) = 1$, $f''(1) = 2$, $f'(0) = 3$, $f'(1) = 4$, $f(0) = 5$, $f(1) = 6$. Compute $\int_0^1 f(x)f'(x)dx$.

19. For some constants A and B, the rate of production $R(t)$ of oil in a new oil well is modelled by:

$$R(t) = A + Be^{-t}\sin(2\pi t),$$

where t is the time in years, A is the equilibrium rate, and B is the"variable" coefficient.

(a) Find the total amount of oil produced in the first N years of operation.

(b) Find the average amount of oil produced per year over the first N years.

(c) From your answer to part (b), find the average amount of oil produced per year as $N \to \infty$.

(d) Looking at the function $R(t)$, explain how you might have predicted your answer to part (c) without doing any calculations.

(e) Do you think it is reasonable to expect this model to hold over a very long period?

20. The rate, r, at which a population of bacteria grows can be modeled by $r = te^{3t}$, where t is time in days. Find the total population of bacteria after 20 days.

21. Recall that the error in the tangent line approximation to $f(x)$ at $x = a$ is given by

$$E(x) = f(x) - f(a) - f'(a)(x - a).$$

(a) Show that $\int_a^x (x - t)f''(t)\mathrm{d}t = E(x)$.

22. Suppose $f(0) = 1$, $f(1) = e$, and $f'(x) = f(x)$ for all x. Find

$$\int_0^1 e^x f'(x)dx.$$