MATH4450: HOMEWORK 10

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Abstract. These notes discuss properties of uniformly convergent sequences of functions.

1. Exercises

Recall the following definition

Definition 1.1. A sequence of functions $f_n : S \to \mathbb{R}$ with $S \subseteq \mathbb{R}$ and $n \in \mathbb{N}$, *converges uniformly* to a function $f : S \to \mathbb{R}$ if for any $\epsilon > 0$, there exists a number N such that

$$|f(x) - f_n(x)| < \epsilon$$

for all $n > N$, and all $x \in S$. We often write $f_n \to f$ uniformly on S.

There is the following lemma:

Lemma 1.2. *Let $f_n : [a, b] \to \mathbb{R}$ be sequence of continuous functions on a closed interval. If $f_n \to f$ uniformly on $[a, b]$, then f is continuous.*

Exercise 1. Prove the previous lemma.

Proof. Fix $\alpha \in [a, b]$. We must show

$$\lim_{x \to \alpha} f(x) = f(\alpha).$$

In other words, given $\epsilon > 0$, we must show there exists a $\delta > 0$ such that

$$|f(\alpha) - f(x)| < \epsilon$$

whenever $|\alpha - x| < \delta$.

First fix $n > 0$ such that $|f(x) - f_n(x)| < \epsilon/3$, for all $x \in [a, b]$; this is possible due to the uniform convergence of the sequence. On the other hand, since f_n is continuous, there exists a $\delta > 0$ such that $|f_n(\alpha) - f_n(x)| < \epsilon/3$ whenever $|\alpha - x| < \delta$. It follows that

$$\begin{aligned} |f(\alpha) - f(x)| &= |f(\alpha) - f_n(\alpha) + f_n(\alpha) - f_n(x) + f_n(x) - f(x)| \\ &\leq |f(\alpha) - f(x)| + |f_n(\alpha) - f_n(x)| + |f_n(x) - f(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}, \end{aligned}$$

whenever $|\alpha - x| < \delta$. $\square$

This allows us to prove the following theorem

Theorem 1.3. *Let $f_n : [a, b] \to \mathbb{R}$ be sequence of continuous functions on a closed interval converging uniformly to f. Then*

$$\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt.$$

Exercise 2. Prove the previous theorem.

Proof. Fix $\epsilon > 0$. Due to the uniform convergence, there exists N such that $|f(t) - f_n(t)| < \epsilon/(b-a)$ for all $n > N$, and all $t \in [a, b]$. It follows that

$$\left| \int_a^b f(t) - \int_a^b f_n(t) \right| = \left| \int_a^b f(t) - f_n(t)dt \right| < \frac{\epsilon}{(b-a)}(b-a),$$

for all $n > N$. □

In other words, continuity, and integration are well behaved for uniformly convergent sequences of functions.

On the other hand, differentiability is not as well behaved. We state without proof the Weierstraß polynomial approximation theorem.

Theorem 1.4 (Weierstraß)**.** *For any continous function $f : [a, b] \to \mathbb{R}$, there exists a sequence of polynomials $p_n(x)$ converging uniformly to f on $[a, b]$.*

Proof. See for instance Browder [1, Theorem 7.1]. □

In other words, there are uniformly convergent sequences of differentiable functions, whose limit is not differentiable.

Moreover:

Lemma 1.5. *There exists a sequence of differentiable functions $f_n : [0, 1] \to \mathbb{R}$ converging uniformly to 0, such that the sequence f_n' does not converge to 0.*

In other words, for uniformly convergent sequences of differentiable functions, even if the limit function is differentiable, it is not necessarily the case that the derivative of the limit is the limit of the derivatives.

Exercise 3. Prove the previous lemma.

[Hint: Consider the functions $f_n(x) = \frac{\sin(nx)}{n}$.]

Proof. Consider the sequence of functions in the hint. Clearly the functions are differentiable. Since

$$|f_n(x)| = |1/n|$$

the sequence converges uniformly to 0.

On the other hand, $f_n'(x) = \cos(nx)$, which does not converge to 0. Indeed for any $n > 0$, $f_n'(0) = 1$. □

There are sequences of functions which are well behaved in all respects. Recall the following definition:

Definition 1.6. A function $f : (a,b) \to \mathbb{R}$ is analytic at $x_0 \in (a,b)$ if there exists a sequence $(a_k)_{k=0}^{\infty}$ and a $\delta > 0$ such that the sequence of functions $f_n(x) = \sum_{k=0}^{n} a_k(x-x_0)^k$ converges to $f(x)$ whenever $|x - x_0| < \delta$.

Theorem 1.7. *Consider the power series* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. *Fix*

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Then the series converges absolutely for $|x - x_0| < R$, *uniformly for* $|x - x_0| \leq R' < R$, *and diverges for* $|x - x_0| > R$.

Proof. Fix k such that

$$\frac{1}{R} < k < \frac{1}{R'}.$$

By our choice of R, for $n >> 0$, we have $|a_n|^{1/n} < k$. It follows that for $|x - x_0| \leq R'$,

$$|a_n(x-x_0)^n| = \left(|a_n|^{1/n}|x-x_0|\right)^n < (kR')^n.$$

Since k was chosen so that $kR' < 1$, and hence $\sum_{n=0}^{\infty}(kR')^n$ converges, it follows from the Weierstraß M-test that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly. This also shows absolute convergence.

To show divergence when $|x - x_0| > R$, choose

$$\frac{1}{|x-x_0|} < k < \frac{1}{R}.$$

From the definition of R, there are an infinite number of a_n such that $|a_n|^{1/n} > k$. For such an a_n,

$$|a_n(x-x_0)^n| = \left(|a_n|^{1/n}|x-x_0|\right)^n > (k|x-x_0|)^n > 1.$$

This violates the principle that the summands in a convergent series must go to zero. □

Remark 1.8. Less can be said for $|x - x_0| = R$. Consider the series $\sum_{n=0}^{\infty} x^n$ which converges at no point where $|x| = 1$, and the series $\sum_{n=0}^{\infty} x^n/n^2$ which converges at every point on $|x| = 1$.

Remark 1.9. Using the ratio test, one can check that

$$R = \frac{1}{\lim_{n\to\infty} |a_{n+1}/a_n|}.$$

Indeed,

$$\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x-x_0|.$$

One can also show (although we omit a proof)

Theorem 1.10. *Let $\sum_{n=0}^\infty a_n(x-x_0)^n$ be a power series with positive radius of convergence R, converging to a function f. Then*

$$\sum_{n=1}^\infty n a_n (x-x_0)^{n-1}$$

has the same radius of convergence, and converges to f'.

Proof. See for example Browder [1, Theorem 4.26]. □

There is also the following result:

Theorem 1.11. *Let $\sum_{n=0}^\infty a_n(x-x_0)^n$ be a power series with positive radius of convergence R, converging to a function f. Then*

$$\sum_{n=0}^\infty \frac{a_n}{n+1}(x-x_0)^{n+1}$$

has the same radius of convergence, and converges to a function F such that $dF/dx = f$.

Exercise 4. Prove the previous theorem. [Hint: use Theorem 1.3].

Proof. On appropriate closed discs, the partial sums

$$S_N = \sum_{n=0}^N a_n(x-x_0)^n$$

converge uniformly to f. Thus by Theorem 1.3,

$$\lim_{N\to\infty} \sum_{n=0}^N \frac{a_n}{n+1}(x-x_0)^{n+1} = \lim_{N\to\infty} \int_{x_0}^x S_N(t)dt = \int_{x_0}^x f(t)dt.$$

Thus the series converges to an anti-derivative of f on the same disc on which the original power series converged. The radius of convergence can not be larger due to the previous theorem. □

The references above are to a book by A. Browder [1]. All of these results are also in the standard book below by W. Rudin [2]. I will try to update the references to this book at some point.

References

1. A. Browder, *Mathematical analysis*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1996, An introduction.
2. W. Rudin, *Principles of mathematical analysis*, third ed., McGraw-Hill Book Co., New York, 1976, International Series in Pure and Applied Mathematics.