# ABSTRACT ALGEBRA 2 PRACTICE EXAM AND HOMEWORK

## 1. Practice exam problems

**Problem A.** Find $\alpha \in \mathbb{C}$ such that $\mathbb{Q}(i, \sqrt[3]{2}) = \mathbb{Q}(\alpha)$.

**Problem B.** Let $\phi_2$ be the Frobenius automorphism of $\mathbb{F}_4$, the field with 4 elements. Let $0, 1, \alpha, \beta$ be the elements of $\mathbb{F}_4$. Describe $\phi_2$ by indicating the image of each element of $\mathbb{F}_4$ under this map (e.g. $\phi_2(0) = 0$).

**Problem C.** Give an example of a degree two field extension that is not Galois.

**Problem D.** Let $\zeta \in \mathbb{C}$ be a primitive 5-th root of unity. Find all field extensions $K$ of $\mathbb{Q}$ contained in $\mathbb{Q}(\zeta)$. For each such field extension, find an element $\alpha \in \mathbb{Q}(\zeta)$ such that $K = \mathbb{Q}(\alpha)$.

**Problem E.** Let $F$ be a field. For a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in F[x]$ we define the derivative $f'(x)$ of $f(x)$ to be the polynomial

$$f'(x) = \sum_{i=1}^{n} i a_i x^{i-1}.$$

(a) Show that the map $D : F[x] \to F[x]$ given by $D(f(x)) = f'(x)$ is a linear map of vector spaces.
(b) Find $\ker(D)$. [Hint: The answer may depend on the characteristic of $F$.]
(c) Show that $D$ satisfies the Leibniz rule: $D(f(x)g(x)) = D(f(x))g(x) + f(x)D(g(x))$ for all $f(x), g(x) \in F[x]$.
(d) Show that $D((f(x)^m)) = mf(x)^{m-1}D(f)$ for each $m \in \mathbb{Z}_{\geq 0}$.

**Problem F.** Let $\bar{F}$ be an algebraic closure of a field $F$. Show that $f(x) \in F[x]$ has a root $\alpha \in \bar{F}$ of multiplicity $\mu > 1$ if and only if $\alpha$ is a root of both $f(x)$ and $f'(x)$. [Hint: Consider the factorization $f(x) = (x - \alpha)^{\mu} g(x)$ in $\bar{F}[x]$ and use the previous problem.]

**Problem G.** Let $F$ be a field, and let $t$ be a variable. Let

$$s = \frac{p(t)}{q(t)} \in F(t).$$

and let $F(s) \hookrightarrow F(t)$ be the associated inclusion of fields. Assuming $s \notin F$, and $p(t)$ and $q(t)$ have no common irreducible factors, show that

$$[F(t) : F(s)] = \max(\deg(p(t)), \deg(q(t))).$$

[Hint: Consider the polynomial $p(X) - sq(X) \in F(s)[X]$ and recall that if $D$ is a UFD with field of fractions $K$, and $f(X) \in D[X]$ is a primitive polynomial, then $f(X)$ is irreducible in $D[X]$ if and only if it is irreducible in $K[X]$.]

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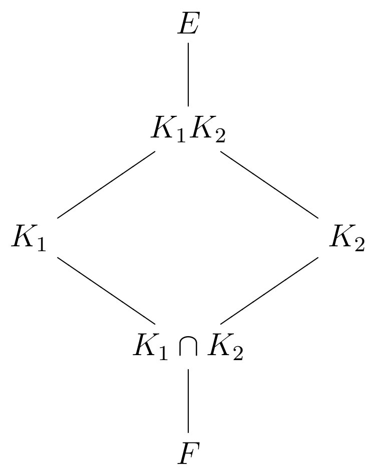
**Problem H.** Let $E/F$ be an extension of fields. Let $K_1, K_2$ be two finite field extensions of $F$ contained in $E$. Show that if $K_1$ is a normal extension of $F$, then $K_1K_2$ is a normal extension of $K_2$.

**Problem I** (Optional)**.** Let $E$ be a finite Galois extension of a field $F$. Let $K_1$ and $K_2$ be two extensions of $F$ contained in $E$. We obtain a diagram of field extensions




Show that $G(E/(K_1K_2)) = G(E/K_1) \cap G(E/K_2) \subseteq G(E/F)$ and $G(E/(K_1 \cap K_2)$ is the subgroup $G$ of $G(E/F)$ generated by the set

$$G(E/K_1)G(E/K_2) = \{\sigma_1\sigma_2 : \sigma_1 \in G(E/K_1), \sigma_2 \in G(E/K_2)\}.$$

[Hint: For the first part, to show $G(E/(K_1K_2)) \supseteq G(E/K_1) \cap G(E/K_2)$, come up with a useful description of the elements of $K_1K_2$ in terms of those in $K_1$ and $K_2$. For the second part, use Galois theory to show $E^G = K_1 \cap K_2$.]

**Problem J.** Let $E/F$ be an extension of fields. Let $K_1, K_2$ be two field extensions of $F$ contained in $E$. If $K_1$ is a finite Galois extension of $F$, then $K_1K_2$ is Galois over $K_2$. Moreover, there is an isomorphism

$$\phi : G(K_1K_2/K_2) \to G(K_1/(K_1 \cap K_2))$$

given by $\sigma \mapsto \sigma|_{K_1}$.

## 2. Homework on $\mathbb{P}GL_2(F)$

**Problem K.** Let $F$ be a field, and let $M_2(F)$ be the set of $2 \times 2$ matrices with entries in $F$. The group of invertible matrices, $GL_2(F)$, is the subset consisting of those matrixes $A \in M_2(F)$ such that $\det(A) \neq 0$. For $\lambda \in F$, we will denote by $[\lambda]$ the matrix entries $\lambda$ on the diagonal, and zeros in every other entry. In other words,

$$[\lambda] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

(1) Show that we may define an equivalence relation on $GL_n(F)$ by setting

$$A \sim A'$$

whenever $A, A' \in GL_n(F)$ and there exists $\lambda \in F^*$ such that $A = [\lambda]A'$.

(2) We define a set $\mathbb{P}GL_2(F)$ to be the quotient of $GL_2(F)$ by this equivalence relation. I.e.

$$\mathbb{P}GL_2(F) = GL_2(F)/\sim .$$

We will use the notation $\bar{A}$ for the equivalence class of a matrix $A \in GL_2(F)$ in $PGL_2(F)$. Show that $\mathbb{P}GL_2(F)$ is a group under the composition law given by $\bar{A}\bar{A}' = \overline{AA'}$.

**Problem L.** Let $F$ be a field. Let $G$ be the subset of $F(x)^*$ consisting of elements of the form

$$\frac{ax+b}{cx+d}$$

such that there does not exist $\lambda \in F^*$ such that $ax+b = \lambda(cx+d)$.

(1) Show that

$$G = \left\{ \frac{ax+b}{cx+d} \in F(x)^* : ad - bc \neq 0 \right\}.$$

(2) Show that $G$ is a group under composition.

(3) Show that there is a group isomorphism

$$G \to \mathbb{P}GL_2(F)$$

given by

$$\frac{ax+b}{cx+d} \mapsto \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}.$$

## 3. Optional homework problems on $\mathbb{P}^1_F$

**Problem M.** Let $F$ be a field.

(1) Show that we may define an equivalence relation on $F^2 - (0,0)$ by setting

$$(x_0, x_1) \sim (x_0', x_1')$$

if and only if there exists $\lambda \in F^*$ such that $(x_0, x_1) = (\lambda x_0', \lambda x_1')$.

(2) We define the projective line over $F$, denoted $\mathbb{P}^1_F$, to be the quotient of $F^2 - (0,0)$ by this equivalence relation. I.e.

$$\mathbb{P}^1_F = \left(F^2 - (0,0)\right) / \sim .$$

We use the notation $[x_0 : x_1]$ for the equivalence class of $(x_0, x_1)$ in $\mathbb{P}^1$. Now let

$$U_0 = \{[x_0 : x_1] \in \mathbb{P}^1_F : x_0 \neq 0.\}$$

Show that there is a bijection of sets

$$F \to U_0 \subset \mathbb{P}^1_F$$

given by $a \mapsto [1 : a]$.

(3) Show that

$$\mathbb{P}^1_F = U_0 \sqcup [0 : 1].$$

In other words, using (2) we can think of the projective line as our field $F$ together with one "extra" point. This point is typically called the point at infinity.

**Problem N.** Let $F$ be a field. A polynomial $f(X_0, X_1) \in F[X_0, X_0]$ is homogeneous of degree $d \in \mathbb{Z}_{\geq 0}$ if each monomial (with non-zero coefficient) in $f(X_0, X_1)$ is of degree $d$. For instance, $X_0^2 - X_0X_1$ is homogeneous of degree 2, whereas $X_0^2 - X_1$ is not homogeneous. In general, we may write a homogeneous polynomial of degree $d$ in the form

$$f(X_0, Y_0) = \sum_{i=0}^{d} a_i X_0^{d-i} X_1^i,$$

for some $a_0, \ldots, a_d \in F$.

(1) Show that if $f(X_0, X_1) \in F[X_0, X_1]$ is homogenous of degree $d$ then for each $\lambda \in F$,

$$f(\lambda X_0, \lambda X_1) = \lambda^d f(X_0, X_n).$$

(2) Use part (1) to show that if $f_0(X_0, X_1)$ and $f_1(X_0, X_1)$ are homogeneous polynomials of degree $d > 0$ with no common roots in $F$, then there is a well defined map of sets

$$f : \mathbb{P}^1_F \to \mathbb{P}^1_F$$

given by $[x_0 : x_1] \mapsto [f_0(x_0, x_1) : f_1(x_0, x_1)]$.

(3) Assume that $F$ is algebraicly closed and $\mathrm{char}(F) = p$. Show that the map in (2) is bijective if and only if $f_0 = (a_0X_0 + b_0X_1)^{p^m}$ and $f_1 = (a_1X_0 + b_1X_1)^{p^m}$ for some integer $m \geq 0$, and some $a_0, a_1, b_0, b_1 \in F$. We use the convention that $0^m = 1$ for all $m$.

**Problem O.** Let $F$ be a field.

(1) Consider the subset

$$F(X_0, X_1)_0 := \left\{ \frac{p(X_0, X_1)}{q(X_0, X_1)} \in F(X_0, X_1) : p, q \in F[X_0, X_1], q \neq 0, \text{and} \right.$$
$$\left. p, q \text{ are homogeneous of the same degree} \right\}$$

Show that this is a subfield of $F(X_0, X_1)$.

(2) Show that there is an isomorphism of fields

$$\Phi : F(x) \to F(X_0, X_1)_0$$

given by

$$\frac{\sum_{i=0}^n a_i x^i}{\sum_{j=0}^m b_j x^j} \mapsto X_0^{m-n} \frac{\sum_{i=0}^n a_i X_0^{n-i} X_1^i}{\sum_{j=0}^m b_j X_0^{m-j} X_1^j}$$