APPENDIX C

Groupoids

This appendix is written for two purposes. It can serve as a reference for facts about categories in which all morphisms are isomorphisms. More importantly, it can be regarded as a short text on groupoids and stacks of discrete spaces. In this way it can provide an introduction to many of the ideas and constructions that are made in the main text, without any algebro-geometric complications.

In this appendix, all categories are assumed to be small. This is not so much for set-theoretic reasons (cf. B §5, but rather to think and write about their objects and morphisms as discrete spaces of points.

If X is a category, we write X_0 for the set of its objects, X_1 for the set of its morphisms and $s, t \colon X_1 \to X_0$ for the source and target map. The notation $a \colon x \to y$ or $x \xrightarrow{a} y$ means that a is in X_1 and $s(a) = x$, $t(a) = y$. The set of morphisms from x to y is denoted $\mathrm{Hom}(x, y)$. The composition, or multiplication, is defined on the collection $X_2 = X_1 \,{}_t{\times}_{X_0}{}_s\, X_1$ of pairs (a, b) such that $t(a) = s(b)$. We write $b \circ a$ or $a{\cdot}b$ for the composition of a and b. We denote by $m \colon X_2 \to X_1$ the map that sends (a, b) to $a{\cdot}b$. There is also a map $e \colon X_0 \to X_1$ that takes every object x to the identity morphism id_x or 1_x on that object. In this appendix we generally denote the category by $X_\bullet$.

Exercise C.1. Show that the axioms for a category are equivalent to the following identities among s, t, m, and e: (i) $s \circ e = \mathrm{id}_{X_0} = t \circ e$; (ii) $s \circ m = s \circ p_1$ and $t \circ m = t \circ p_2$, where p_1 and p_2 are the projections from $X_1 \,{}_t{\times}_s\, X_1$ to X_1; (iii) $m \circ (m, 1) = m \circ (1, m)$ as maps from $X_1 \,{}_t{\times}_s\, X_1 \,{}_t{\times}_s\, X_1$ to X_1; (iv) $m \circ (s \circ e, 1) = \mathrm{id}_{X_1} = m \circ (1, t \circ e)$.

We pick a canonical one-element set and denote it pt.

1. Groupoids

Definition C.1. A category $X_\bullet$ is called a **groupoid** if every morphism $a \in X_1$ has an inverse. There exists therefore a map $i \colon X_1 \to X_1$ that takes a morphism to its inverse. The element $i(a)$ is often denoted a^{-1}.

Exercise C.2. A groupoid is a pair of sets X_0 and X_1, together with five maps s, t, m, e and i, satisfying the four identities of the preceding exercise, together with: (v) $s \circ i = t$ and $t \circ i = s$; (vi) $m \circ (1, i) = e \circ s$ and $m \circ (i, 1) = e \circ t$. Deduce from these identities the properties: (vii) $i \circ i = \mathrm{id}_{X_1}$; (viii) $i \circ e = e$; $m \circ (e, e) = e$; (ix) $i \circ m = m \circ (i \circ p_2, i \circ p_1)$. Show that e and i are uniquely determined by X_0, X_1, s, t, and m.

We will generally think of a groupoid $X_\bullet$ as a pair of sets (or discrete spaces) X_0 and X_1, with morphisms s, t, m, e, and i, satisfying these identities. Occasionally, however, we will use the categorical language, referring to elements of X_0 as *objects* and elements of X_1 as *arrows* or *morphisms*. The notation $X_1 \rightrightarrows X_0$ may be used in place of $X_\bullet$.

Definition C.2. For any $x \in X_0$, the composition m defines a group structure on the set $\mathrm{Hom}(x,x) = \{a \in X_1 \mid s(a) = x, t(a) = x\}$. This group is denoted $\mathrm{Aut}(x)$, and it is called the **automorphism** or **isotropy** group of x.

A groupoid may be thought of as an approximation of a group, but where composition is not always defined.

Our first example is the prototype groupoid:

Example C.3. Let X be a topological space. Define the **fundamental groupoid** $\pi(X)_\bullet$ by taking $\pi(X)_0 = X$ as the set of objects and

$$\pi(X)_1 = \{\gamma\colon [0,1] \to X \text{ continuous}\}/\sim$$

as the set of arrows. Here we write $\gamma \sim \gamma'$ for two paths in X if there exists a homotopy between γ and γ' fixing the endpoints. Then we define

$$s\colon \pi(X)_1 \longrightarrow \pi(X)_0, \qquad [\gamma] \longmapsto \gamma(0)$$

and

$$t\colon \pi(X)_1 \longrightarrow \pi(X)_0 \qquad [\gamma] \longmapsto \gamma(1).$$

Thus the paths γ and γ' are composable precisely if $\gamma(1) = \gamma'(0)$ and we have

$$\pi(X)_2 = \{([\gamma],[\gamma']) \in \pi(X)_1 \times \pi(X)_1 \mid \gamma(1) = \gamma'(0)\}.$$

The composition of $[\gamma]$ and $[\gamma']$ is defined to be the homotopy class of the path

$$(\gamma{\cdot}\gamma')(t) = \begin{cases} \gamma(2t) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1) & \text{if} \quad \frac{1}{2} \le t \le 1 \end{cases},$$

$$\underset{\gamma\cdot\gamma'}{\bullet \xrightarrow{\gamma} \bullet \xrightarrow{\gamma'} \bullet}$$

[There should be a nicely drawn picture of paths here.] Thus we have

$$m\colon \pi(X)_2 \longrightarrow \pi(X)_1, \qquad ([\gamma],[\gamma']) \longmapsto [\gamma{\cdot}\gamma'].$$

Exercise C.3. Prove that $\pi(X)_\bullet$ is a groupoid. In particular, determine the maps $e\colon \pi(X)_0 \to \pi(X)_1$ and $i\colon \pi(X)_1 \to \pi(X)_1$. More generally, for any subset A of X, construct a groupoid $\pi(X,A)_\bullet$, with $\pi(X,A)_0 = A$ and $\pi(X,A)_1$ the set of homotopy classes of paths with both endpoints in A.

It is useful to imagine any groupoid geometrically in terms of paths as suggested by this example. (It is in examples like this that the notation $a{\cdot}b$ is preferrable to the $b \circ a$ convention.)

The fundamental mathematical notions of *set* and *group* occur as extreme cases of groupoids:

Example C.4. Every set X is a groupoid by taking the set of objects X_0 to be X and allowing only identity arrows, which amounts to taking $X_1 = X$, too. We consider every set as a groupoid in this way, if not mentioned otherwise.

Example C.5. Every group G is a groupoid by taking $X_0 = pt$ and declaring the automorphism group of the unique object of $X_\bullet$ to be G. Then $\mathrm{Aut}(x) = G = X_1$, if x denotes the unique element of pt. In this appendix we write $X_\bullet = BG_\bullet$ and call it the **classifying groupoid** of the group G.

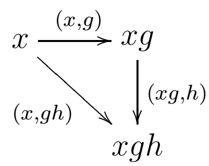
The next example contains the previous two. It describes a much more typical groupoid:

Example C.6. If X is a right G-set, we define a groupoid $X \rtimes G$ by taking X as the set of objects of $X \rtimes G$ and declaring, for any two $x, y \in X$,

$$\mathrm{Hom}(x, y) = \{g \in G \mid xg = y\}.$$

Composition in $X \rtimes G$ is induced from multiplication in G.

More precisely, we have $(X \rtimes G)_0 = X$ and $(X \rtimes G)_1 = X \times G$. The source map $s\colon X \times G \to X$ is the first projection, the target map $t\colon X \times G \to X$ is the action map: $t(x, g) = xg$. The morphisms (x, g) and (y, h) are composable if and only if $y = xg$ and the multiplication is given by $(x, g)\boldsymbol{\cdot}(y, h) = (x, gh)$:



Thus we may identify X_2 with $X \times G \times G$, with $(x, g) \times (xg, h)$ corresponding to (x, g, h), and write

$$m\colon X \times G \times G \longrightarrow X \times G, \quad (x, g, h) \longmapsto (x, gh).$$

The groupoid $X \rtimes G$ is called the **transformation groupoid** given by the G-set X.

Example C.7. If X is a left G-set, we get an associated groupoid by declaring

$$\mathrm{Hom}(x, y) = \{g \in G \mid gx = y\}.$$

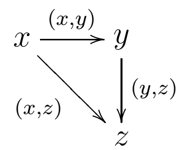
Thus the pair (g, x) is considered as an arrow from x to gx. The source map is again the projection and the target map is the group action. We denote this groupoid by $G \ltimes X$. Note that the multiplication is given by $(g, x)\boldsymbol{\cdot}(h, gx) = (hg, x)$, which reverses the order of the group elements.[1]

Exercise C.4. Suppose a set X has a left action of a group G and a right action of a group H, and these actions **commute**, i.e., $(gx)h = g(xh)$ for all $g \in G, x \in X, h \in H$; in this case we write gxh for this common element. Construct a **double transformation groupoid** $G \ltimes X \rtimes H$, of the form $G \times X \times H \rightrightarrows X$, with $s(g, x, h) = x$, $t(g, x, h) = gxh$, and $m((g, x, h), (g', gxh, h')) = (g'g, x, hh')$.

The next two examples go beyond group actions on sets:

[1] This notation is compatible with the composition notation $b \circ a$, which is useful in the common situation where an automorphism group of a mathematical structure is considered to act on the left, with the product given by composition.

EXAMPLE C.8. If $R \subset X \times X$ is an equivalence relation on the set X, then we define an associated groupoid $R \rightrightarrows X$ by taking the two projections as source and target map: $s = p_1$, $t = p_2$. Composition is given by $(x, y)\cdot(y, z) = (x, z)$:



For $x, y \in X$ there is at most one morphism from x to y and x and y are isomorphic in the groupoid $R \rightrightarrows X$ (meaning that there is an a in $X_1 = R$ with $s(a) = x$ and $t(a) = y$) if and only if $(x, y) \in R$, i.e., x and y are equivalent under the relation R.

EXAMPLE C.9. Let $(G_i)_{i\in I}$ be a family of groups. Define an associated groupoid by taking as objects the set $X_0 = I$. We declare all objects to be pairwise non-isomorphic and define, for each $i \in I$, $\mathrm{Aut}(i) = G_i$. Then X_1 is the disjoint union $\coprod_{i\in I} G_i$ and $s = t$ maps $g \in G_i$ to i.

EXAMPLE C.10. More generally, if $(X_\bullet(i))_{i\in I}$ is any family of groupoids, there is a **disjoint union** groupoid $X_\bullet = \coprod_i X_\bullet(i)$, with $X_0 = \coprod_i X_0(i)$ and $X_1 = \coprod_i X_1(i)$.

EXAMPLE C.11. Let $X_0 \to Y$ be any map of sets. Define an associated groupoid $X_\bullet$ by defining X_1 to be the fibered product: $X_1 = X_0 \times_Y X_0$. The source is the first projection and the target is the second projection. Call this groupoid the **cross product groupoid** associated to $X_0 \to Y$. Note that this construction is a special case of an equivalence relation (Example C.8).

EXAMPLE C.12. For any set X, there is a groupoid with $X_0 = X$, and $X_1 = X \times X$, with s and t the two projections, and $(x, y)\cdot(y, z) = (x, z)$. This is also an equivalence relation, with any two points being equivalent. This is sometimes called a **banal** groupoid. It is a special case of the preceding example, with $Y = pt$.

DEFINITION C.13. Given a groupoid $X_\bullet$, a **subgroupoid** is given by subsets $Y_0 \subset X_0$ and $Y_1 \subset X_1$ such that: $s(Y_1) \subset Y_0$; $t(Y_1) \subset Y_0$; $e(Y_0) \subset Y_1$, $i(Y_1) \subset Y_1$, and $a, b \in Y_1$ with $t(a) = s(b)$ implies $a\cdot b \in Y_1$.

EXERCISE C.5. Let Z be any set. Construct a groupoid with X_0 the set of nonempty subsets of Z, and with $X_1 = \{(A, B, \phi) \mid A, B \in X_0 \text{ and } \phi\colon A \to B \text{ is a bijection}\}$, and multiplication given by $(A, B, \phi)\cdot(B, C, \psi) = (A, C, \psi \circ \phi)$.

EXERCISE C.6. Let Γ be a directed graph, which consists of a set V (of vertices) and a set E of edges, together with mappings $s, t\colon E \to V$. For any $a \in E$, define a symbol $\widetilde{a}$, called the *opposite edge* of a, and set $s(\widetilde{a}) = t(a)$ and $t(\widetilde{a}) = s(a)$. For each $v \in V$ define a symbol 1_v, with $s(1_v) = t(1_v) = v$. Construct a groupoid $F(\Gamma)_\bullet$, called the *free groupoid* on Γ, by setting $F(\Gamma)_0 = V$, and $F(\Gamma)_1$ is the (disjoint) union of $\{1_v \mid v \in V\}$ and the set of all sequences $(\alpha_1, \ldots, \alpha_n)$, with each α_i either an edge or an opposite edge, with $t(\alpha_i) = s(\alpha_{i+1})$, such that no successive pair (α_i, α_{i+1}) has the form $(a, \widetilde{a})$ or $(\widetilde{a}, a)$ for any edge a, $1 \le i < n$. Composition is defined by juxtaposition and canceling to eliminate successive pairs equal to an edge and its inverse. Verify that $F(\Gamma)_\bullet$ is a groupoid.

Exercise C.7. Let $X_\bullet$ be a groupoid in which the multiplication map $m\colon X_2 \to X_1$ is finite-to-one. For any commutative ring K with unity, let $A = K[X_\bullet]$ be the set of K-valued functions on X_1. Define a *convolution product* on A by the formula

$$(f * g)(c) = \sum_{a\bullet b=c} f(a)\cdot f(b),$$

the sum over all pairs $a, b \in X_1$ with $a\bullet b = c$. Show that, with the usual pointwise sum for addition, this makes A into an associative K-algebra with unity. If $X_\bullet = BG_\bullet$, this is the group algebra of G. (Extending this to infinite groupoids, with appropriate measures to replace the sums by integrals, is an active area (cf. [**18**]), as it leads to interesting $\mathbb{C}^*$-algebras.)

Remark C.14. There is an obvious notion of isomorphism between groupoids $X_\bullet$ and $Y_\bullet$. It is given by a bijection between X_0 and Y_0 and a bijection between X_1 and Y_1, compatible with the structure maps s, t, m (and therefore e and i). This notion will be referred to as **strict isomorphism**, since it is too strong for most purposes. We will define a more supple notion of isomorphism in the next section.

Exercise C.8. Any left action of a groups G on a set X determines a right action of G on X by setting $x\cdot g = g^{-1}x$. Show that the map which is the identity on X, and maps $G\times X$ to $X\times G$ by $(g,x)\mapsto (x,g^{-1})$, determines a strict isomorphism of $G \ltimes X$ with $X \rtimes G$.

Exercise C.9. Let $X_\bullet$ be a groupoid. Define the groupoid $\widetilde{X}_\bullet$ by reversing the direction of arrows. In other words, $\widetilde{X}_0 = X_0$, $\widetilde{X}_1 = X_1$, $\tilde{s} = t$, $\tilde{t} = s$, $\widetilde{X}_2 = \{(x,y) \in X_1\times X_1 \mid (y,x)\in X_2\}$ and $\widetilde{m}(x,y) = m(y,x)$. This is a groupoid (with $\tilde{e} = e$ and $\tilde{i} = i$). Show that $\widetilde{X}_\bullet$ is strictly isomorphic to $X_\bullet$ by sending an element of X_1 to its inverse, and the identity on X_0. This is called the **opposite groupoid** of $X_\bullet$, and is often denoted $X_\bullet^{\mathrm{opp}}$.

Exercise C.10. For a left action of a group G on a set X, define a groupoid with $X_0 = X$, $X_1 = G\times X$, with $s(g,x) = g\cdot x$, $t(g,x) = x$, and $m((g,h\cdot x),(h,x)) = (h\cdot g, x)$. Show that this is a groupoid, strictly isomorphic to the opposite groupoid of $G\ltimes X$. Similarly for a right action of G on X, there is a groupoid with $X_0 = X$, $X_1 = X\times G$, with $s(x,g) = x\cdot g$, $t(s,g) = x$, and $(x\cdot h, g)\bullet(x,h) = (x, h\cdot g)$; this is strictly isomorphic to the opposite groupoid of $X\rtimes G$.

The preceding exercises show that, although there are several possible conventions for constructing transformation groupoids of actions of a group on a set, they all give strictly (and canonically) isomorphic groupoids.

Exercise C.11. By a **right action** of a group G on a groupoid $X_\bullet$ is meant a right action of G on X_1 and on X_0, so that s, t are equivariant[2], and satisfying $ag\bullet bg = (a\bullet b)g$

[2]A mapping $f\colon U\to V$ of right G-sets is **equivariant** if $f(ug) = f(u)g$ for all $u\in U$ and $g\in G$.

for $a, b \in X_1$ with $t(a) = s(b)$, and $g \in G$; that is, m is equivariant with repect to the diagonal action on X_2. It follows that e and i are also equivariant. Construct a groupoid $X_1 \times G \rightrightarrows X_0$, denoted $X_\bullet \rtimes G$, by defining $s(a, g) = s(a)$, $t(a, g) = t(ag) = t(a)g$, and $(a, g)\cdot(b, g') = (a\cdot bg^{-1}, gg')$. Verify that $X_\bullet \rtimes G$ is a groupoid. Construct a groupoid $G \ltimes X_\bullet$ for a left action.

Exercise C.12. Suppose a groupoid $X_\bullet$ has a left action of a group G, and a right action of a group H, and the actions commute, i.e., $(gx)h = g(xh)$ for $g \in G$, $h \in H$, and $x \in X_0$ or X_1. There is a natural right action of H on $G \ltimes X_\bullet$, and a left action of G on $X \rtimes H$. Construct a strict isomorphism between the groupoids $(G \ltimes X_\bullet) \rtimes H$ and $G \ltimes (X_\bullet \rtimes H)$.

Exercise C.13. (*)[3] For every groupoid $X_\bullet$, construct a topological space X and a subset A so that $X_\bullet$ is strictly isomorphic to the fundamental groupoid $\pi(X, A)_\bullet$.

Let us consider two basic properties of groupoids:

Definition C.15. A groupoid $X_\bullet$ is called **rigid** if for all $x \in X_0$ we have $\mathrm{Aut}(x) = \{\mathrm{id}_x\}$.

A groupoid $X_\bullet$ is called **transitive** if for all $x, y \in X_0$ there is an $a \in X_1$ with $s(a) = x$ and $t(a) = y$.

Exercise C.14. For a topological space X, $\pi(X)_\bullet$ is rigid if and only if every closed path in X is homotopic to a trivial path, and $\pi(X)_\bullet$ is transitive if and only if X is path-connected.

Exercise C.15. For group actions, the transformation groupoid is rigid exactly when the action is free, and the groupoid is transitive when the action is transitive.

Exercise C.16. Show that every equivalence relation is rigid. Conversely, every rigid groupoid is strictly isomorphic to an equivalence relation.

Definition C.16. A groupoid is canonically and strictly isomorphic to a disjoint union of transitive groupoids, called its **components**. Call two points x and y of X_0 **equivalent** if there is some $a \in X_1$ with $s(a) = x$ and $t(z) = y$, and write $x \cong y$ if this is the case. This is an equivalence relation, defined by the image of X_1 in $X_0 \times X_0$ by the map (s, t). There is a component for each equivalence class; write $X_0/\cong$ for the set of equivalence classes.

Exercise C.17. If $s(a) = x$ and $t(a) = y$, the map $g \mapsto a^{-1}\cdot g\cdot a$ determines an isomorphism from $\mathrm{Aut}(x)$ to $\mathrm{Aut}(y)$. Replacing a by another a' with $s(a') = x$ and $t(a') = y$ gives another isomorphism from $\mathrm{Aut}(x)$ to $\mathrm{Aut}(y)$ that differs from the first by an inner automorphism. Hence there is a group, well-defined up to inner automorphism, associated to each equivalence class of a groupoid: the automorphism group $\mathrm{Aut}(x)$ of any of its points.

Exercise C.18. The free groupoid of a graph is rigid if and only if the graph has no loops. It is transitive when the graph is connected.

[3] The (∗) means that this is a more difficult exercise, which isn't central to understanding.

Next we show how to count in groupoids.

Definition C.17. A groupoid $X_\bullet$ is called *finite* if:

(1) the set of equivalence classes $X_0/\cong$ is finite;
(2) for every object $x \in X_0$ the automorphism group $\mathrm{Aut}(x)$ is finite.

If $X_\bullet$ is a finite groupoid, we define its **mass** to be

$$\#X_\bullet = \sum_{x \in X_0/\cong} \frac{1}{\#\,\mathrm{Aut}(x)},$$

where the sum is taken over a set of representatives of the objects modulo isomorphism. More generally, if each $\mathrm{Aut}(x)$ is finite, and the sums $\sum \frac{1}{\#\,\mathrm{Aut}(x)}$ have a least upper bound, as x varies over representatives of finite subsets of $X_0/\cong$, define the mass $\#X_\bullet$ to be this least upper bound, and call $X_\bullet$ **tame**.

Exercise C.19. Show that if G is a finite group and X a finite G-set, then $X \rtimes G$ is finite and

$$\#X \rtimes G = \frac{\#X}{\#G}.$$

Exercise C.20. $(*)$ Let F be a finite field with q elements. Consider the groupoid $X_\bullet$ of vector bundles over $\mathbb{P}^1_F$ which are of rank 2 and degree 0. The objects of this groupoid are all such vector bundles, the morphisms are all isomorphisms of these vector bundles. Show that this groupoid is tame but not finite, and find its mass.

Definition C.18. A **vector bundle** E on a groupoid $X_\bullet$ assigns to each $x \in X_0$ a vector space E_x, and to each $a \in X_1$ from x to y a linear isomorphism $a_*\colon E_x \to E_y$, satisfying the compatibility: for all $(a,b) \in X_2$, $(a{\cdot}b)_* = b_* \circ a_*$, i.e., with $z = t(b)$, the diagram

$$\begin{array}{ccc} E_x & \xrightarrow{a_*} & E_y \\ & \searrow^{(a\cdot b)_*} & \downarrow{\scriptstyle b_*} \\ & & E_z \end{array}$$

commutes. For example, a vector bundle on $BG_\bullet$ is the same as a representation of the group G.

Exercise C.21. If E is a vector bundle on $X_\bullet$, construct a groupoid $E_\bullet$ with $E_0 = \coprod_{x \in X_0} E_x$, and $E_1 = \{(a,v,w) \mid a \in X_1, v \in E_{sa}, w \in E_{ta}, a_*(v) = w\}$.

2. Morphisms of groupoids

Definition C.19. A **morphism** of groupoids $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is a pair of maps $\phi_0\colon X_0 \to Y_0$, $\phi_1\colon X_1 \to Y_1$, compatible with source, target and composition. In the language of categories, this is the same as a functor.

Example C.20. A continuous map of topological spaces $f\colon X \to Y$ gives rise to a morphism of fundamental groupoids

$$\pi(f)_\bullet\colon \pi(X)_\bullet \longrightarrow \pi(Y)_\bullet \quad .$$

EXAMPLE C.21. Let X and Y be sets. Then the set maps from X to Y are the same as the groupoid morphisms from X to Y.

EXAMPLE C.22. If G and H are groups, then the groupoid morphisms $BG_\bullet \to BH_\bullet$ are the group homomorphisms $G \to H$.

EXAMPLE C.23. Let X be a right G-set and Y a right H-set. Then a morphism $X \rtimes G \to Y \rtimes H$ is given by a pair (ϕ, ψ), where $\phi\colon X \to Y$ and $\psi\colon X \times G \to H$, such that:

(i) for all $x \in X$ and $g \in G$, $\phi(x)\psi(x,g) = \phi(xg)$;
(ii) for all $x \in X$ and g and g' in G, $\psi(x,g)\psi(xg,g') = \psi(x,gg')$.

The pair (ϕ, ψ) induces a groupoid morphism $X \rtimes G \to Y \rtimes H$ by $\phi\colon X \to Y$ on objects and

$$X \times G \longrightarrow Y \times H, \qquad (x,g) \longmapsto (\phi(x), \psi(x,g))$$

on arrows. Every groupoid morphism $X \rtimes G \to Y \rtimes H$ comes about in this way. In particular, if $\rho\colon G \to H$ is a group homomorphism, and $\phi\colon X \to Y$ is *equivariant* with respect to ρ (i.e., $\phi(xg) = \phi(x)\rho(g)$ for $x \in X$ and $g \in G$), then (ϕ, ψ) defines a morphism of groupoids, where $\psi(x,g) = \rho(g)$ for $x \in X$, $g \in G$.

For example, for any right G-set X, the map from X to a point determines a morphism from $X \rtimes G$ to $BG_\bullet$.

EXERCISE C.22. A morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ determines a mapping $X_0/\cong \,\to Y_0/\cong$ of equivalence classes. It also determines a group homomorphism $\mathrm{Aut}(x) \to \mathrm{Aut}(\phi_0(x))$ for every $x \in X_0$, taking a to $\phi_1(a)$.

EXERCISE C.23. If $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is a morphism, and E is a vector bundle on $Y_\bullet$, construct a pullback vector bundle $\phi_\bullet^*(E)$ on $X_\bullet$.

EXERCISE C.24. If $X_\bullet$ and $Y_\bullet$ are equivalence relations, any map $f\colon X_0 \to Y_0$ satisfying $x \sim y \Rightarrow f(x) \sim f(y)$ determines a morphism of groupoids $X_\bullet \to Y_\bullet$, and every morphism from $X_\bullet$ to $Y_\bullet$ arises from a unique such map.

EXAMPLE C.24. If a group G acts (on the right) on a set X, there is a canonical morphism $\pi\colon X \to X \rtimes G$ from the (groupoid of the set) X to the transformation groupoid.

EXERCISE C.25. Let $F(\Gamma)_\bullet$ be the free groupoid on a graph Γ, as in Exercise C.6. For any groupoid $X_\bullet$, show that any pair of maps $V \to X_0$ and $E \to X_1$ comuting with s and t determines a morphism of groupoids from $F(\Gamma)_\bullet$ to $X_\bullet$.

EXERCISE C.26. If $X_\bullet$ and $Y_\bullet$ are groupoids, their (direct) **product** $X_\bullet \times Y_\bullet$ has objects $X_0 \times Y_0$ and arrows $X_1 \times Y_1$, with s, t, and m defined component-wise. More generally, if $X(i)_\bullet$ is a family of groupoids, one has a product groupoid $\prod X(i)_\bullet$.

Of course, morphisms of groupoids may be composed, and we get in this way a category of groupoids (with isomorphisms being the strict isomorphisms considered above). But this point of view is too narrow. In the next section we shall enlarge this category of groupoids to a 2-category.

Exercise C.27. Given morphisms $X_\bullet \to Z_\bullet$ and $Y_\bullet \to Z_\bullet$ of groupoids, construct a groupoid $V_\bullet$ with $V_0 = X_0 \times_{Z_0} Y_0$ and $V_1 = X_1 \times_{Z_1} Y_1$. Show that this is a fibered product in the category of groupoids. (This will *not* be the fibered product in the 2-category of groupoids.)

Exercise C.28. If X is a set and $Y_\bullet$ is a groupoid, a morphism from X to $Y_\bullet$ is given by a mapping of sets from X to Y_0. A morphism from $Y_\bullet$ to X is given by a mapping of sets from $Y_0/\cong$ to X. In categorical language, the functor from (Set) to (Gpd) that takes a set to its groupoid has a right adjoint from (Gpd) to (Set) that takes $Y_\bullet$ to Y_0, and it has a left adjoint from (Gpd) to (Set) that takes $Y_\bullet$ to $Y_0/\cong$.

3. 2-Isomorphisms

Definition C.25. Let $\phi_\bullet$ and $\psi_\bullet$ be morphisms of groupoids from $X_\bullet$ to $Y_\bullet$. A **2-isomorphism** from $\phi_\bullet$ to $\psi_\bullet$ is a mapping $\theta\colon X_0 \to Y_1$ satisfying the following properties:

(1) for all $x \in X_0$: $s(\theta(x)) = \phi_0(x)$ and $t(\theta(x)) = \psi_0(x)$;
(2) for all $a \in X_1$: $\theta(s(a))\cdot\psi_1(a) = \phi_1(a)\cdot\theta(t(a))$.

If $x \xrightarrow{a} y$, we therefore have a commutative diagram

$$\begin{array}{ccc} \phi_0(x) & \xrightarrow{\phi_1(a)} & \phi_0(y) \\ {\scriptstyle\theta(x)}\downarrow & & \downarrow{\scriptstyle\theta(y)} \\ \psi_0(x) & \xrightarrow{\psi_1(a)} & \psi_0(y) \end{array}$$

In the language of categories, this says exactly that θ is a natural isomorphism from the functor $\phi_\bullet$ to the functor $\psi_\bullet$. We write $\theta\colon \phi_\bullet \Rightarrow \psi_\bullet$ to mean that θ is a 2-isomorphism from $\phi_\bullet$ to $\psi_\bullet$.

Example C.26. Consider two continuous maps $f, g\colon X \to Y$ of topological spaces and the groupoid morphisms $\pi(f)_\bullet, \pi(g)_\bullet\colon \pi(X)_\bullet \to \pi(Y)_\bullet$ they induce. Every homotopy $H\colon X \times [0,1] \to Y$ from f to g induces a 2-isomorphism $\pi(H)\colon \pi(f)_\bullet \Rightarrow \pi(g)_\bullet$, which assigns to x in X the homotopy class of the path $t \mapsto H(x,t)$ in Y.

Exercise C.29. Verify that this is a 2-isomorphism from $\pi(f)_\bullet$ to $\pi(g)_\bullet$.

Definition C.27. For a groupoid morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ define the 2-isomorphism $1_{\phi_\bullet}\colon \phi_\bullet \Rightarrow \phi_\bullet$ by $x \mapsto e(\phi_0(x))$ from X_0 to Y_1. For $\phi_\bullet, \psi_\bullet, \chi_\bullet$ from $X_\bullet$ to $Y_\bullet$, and $\alpha\colon \phi_\bullet \Rightarrow \psi_\bullet$ and $\beta\colon \psi_\bullet \Rightarrow \chi_\bullet$, define $\beta \circ \alpha\colon \phi_\bullet \Rightarrow \chi_\bullet$ by the formula $x \to \alpha(x)\cdot\beta(x)$.

Exercise C.30. Show that these definitions define 2-morphisms. Prove that composition is associative, the identities defined behave as identities with respect to composition of 2-isomorphisms, and that every 2-isomorphism is invertible. Conclude that for given groupoids $X_\bullet$ and $Y_\bullet$ the morphisms from $X_\bullet$ to $Y_\bullet$ together with the 2-isomorphisms between them form a groupoid, denoted

$$\mathrm{HOM}(X_\bullet, Y_\bullet).$$

Example C.28. The only 2-isomorphisms between set maps are identities. For sets X, Y, the groupoid $\mathrm{HOM}(X,Y)$ is the set $\mathrm{Hom}(X,Y)$ of maps from X to Y.

Exercise C.31. If Y is a set, then $\mathrm{HOM}(X_\bullet, Y)$ is strictly isomorphic to the set $\mathrm{Hom}(X_0/\cong, Y)$ of maps from $X_0/\cong$ to Y. If $Y_\bullet$ is rigid, then $\mathrm{HOM}(X_\bullet, Y_\bullet)$ is also rigid. If X is a set, then $\mathrm{HOM}(X, Y_\bullet)$ is strictly isomorphic to the groupoid $U_\bullet$ with U_0 the set of maps from X to Y_0 and U_1 the set of maps from X to Y_1.

In particular, for any groupoid $X_\bullet$ there is a canonical morphism

$$\pi\colon X_0 \to X_\bullet$$

from the set X_0 to the groupoid $X_\bullet$. Although this map can be regarded as an inclusion, we will see that it acts more like a projection. There is also a canonical morphism, called the *canonical map*,

$$\rho\colon X_\bullet \to X_0/\cong$$

from the groupoid $X_\bullet$ to the set $X_0/\cong$.

Exercise C.32. Let $X_\bullet$ and $Y_\bullet$ be equivalence relations and $f_\bullet, g_\bullet\colon X_\bullet \to Y_\bullet$ morphisms, given by $f_0, g_0\colon X_0 \to Y_0$. There exists a 2-isomorphism $\theta\colon f_\bullet \Rightarrow g_\bullet$ if and only if $f_0(x) \sim g_0(x)$ for all $x \in X_0$, and such a 2-isomorphism is unique if it exists. It follows that the groupoid $\mathrm{HOM}(X_\bullet, Y_\bullet)$ is an equivalence relation, whose set of equivalence classes has a canonical bijection with the set of maps from $X_0/\cong$ to $Y_0/\cong$.

Example C.29. Let G and H be groups, $\phi, \psi\colon G \to H$ group homomorphisms. Denote by $\phi_\bullet$ and $\psi_\bullet$ the associated morphisms of groupoids $BG_\bullet \to BH_\bullet$. The 2-isomorphisms from $\phi_\bullet$ to $\psi_\bullet$ are the elements $h \in H$ satisfying $\psi(g) = h^{-1}\phi(g)h$, for all $g \in G$.

The groupoid $\mathrm{HOM}(BG_\bullet, BH_\bullet)$ is strictly isomorphic to the transformation groupoid $\mathrm{Hom}(G,H) \rtimes H$, where H acts on the group homomorphisms from G to H by conjugation $(\phi \cdot h)(g) = h^{-1}\phi(g)h$.

Example C.30. Given a G-set X and an H-set Y, and two morphisms (ϕ, ψ) and (ϕ', ψ') from $X \rtimes G$ to $Y \rtimes H$, as in Exercise C.23, a 2-isomorphism from the former to the latter is a map $\theta\colon X \to H$ satisfying: (i) $\phi'(x) = \phi(x)\theta(x)$ for all $x \in X$; (ii) $\psi'(x,g) = \theta(x)^{-1}\psi(x,g)\theta(xg)$ for all $x \in X$ and $g \in G$. In the equivariant case, where $\psi(x,g) = \rho(g)$ and $\psi'(x,g) = \rho'(g)$ for group homomorphisms ρ and ρ' from G to H, the second condition becomes $\rho'(g) = \theta(x)^{-1}\rho(g)\theta(x)$ for all x and g. Show that (ϕ, ψ) is 2-isomorphic to an equivariant map exactly when there is a map $\theta\colon X \to H$ such that for all $g \in G$, the map $x \mapsto \theta(x)^{-1}\psi(x,g)\theta(xg)$ is independent of $x \in X$. [Are there cases where every morphism $X \rtimes G \to Y \rtimes H$ is 2-isomorphic to an equivariant map?]

Exercise C.33. We have seen that a morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ determines a homomorphism from $\mathrm{Aut}(x)$ to $\mathrm{Aut}(\phi_0(x)$ for every $x \in X_0$. A 2-isomorphism $\theta\colon \phi_\bullet \Rightarrow \psi_\bullet$ determines an isomorphism $\mathrm{Aut}(\phi_0(x)) \to \mathrm{Aut}(\psi_0(x))$, taking g to $\theta(x)^{-1}{\cdot}g{\cdot}\theta(x)$. This gives a commutative diagram

$$\begin{array}{ccc} & & \mathrm{Aut}(\phi_0(x)) \\ & \nearrow & \downarrow \\ \mathrm{Aut}(x) & & \\ & \searrow & \\ & & \mathrm{Aut}(\psi_0(x)) \end{array}$$

Definition C.31. Given $\phi_\bullet, \phi'_\bullet \colon X_\bullet \to Y_\bullet$, $\alpha \colon \phi_\bullet \Rightarrow \phi'_\bullet$, and $\psi_\bullet, \psi'_\bullet \colon Y_\bullet \to Z_\bullet$, $\beta \colon \psi_\bullet \Rightarrow \psi'_\bullet$, there is a 2-isomorphism $\beta * \alpha$ from $\psi_\bullet \circ \phi_\bullet$ to $\psi'_\bullet \circ \phi'_\bullet$, that maps x in X_0 to

$$\psi_1(\alpha(x)){\cdot}\beta(\phi'_0(x)) = \beta(\phi_0(x)){\cdot}\psi'_1(\alpha(x))$$

in Z_1.

Exercise C.34. Verify that this defines a 2-isomorphism as claimed. Verify that groupoids, morphisms, and 2-isomorphisms form a 2-category, i.e., that the axioms of Appendix B, §2 are satisfied.

Exercise C.35. Let $I_\bullet$ be the banal groupoid $\{0,1\} \times \{0,1\} \rightrightarrows \{0,1\}$. For any groupoids $X_\bullet$ and $Y_\bullet$, construct a bijection between the morphisms

$$X_\bullet \times I_\bullet \longrightarrow Y_\bullet$$

and the triples $(\phi_\bullet, \psi_\bullet, \theta)$, where $\phi_\bullet$ and $\psi_\bullet$ are morphisms from $X_\bullet$ to $Y_\bullet$ and θ is a 2-isomorphism from $\phi_\bullet$ to $\psi_\bullet$.

4. Isomorphisms

Definition C.32. A morphism of groupoids $\phi_\bullet \colon X_\bullet \to Y_\bullet$ is an **isomorphism of groupoids** if there exists a morphism $\psi_\bullet \colon Y_\bullet \to X_\bullet$, such that $\psi_\bullet \circ \phi_\bullet \cong \mathrm{id}_{X_\bullet}$ and $\phi_\bullet \circ \psi_\bullet \cong \mathrm{id}_{Y_\bullet}$, where '$\cong$' means the existence of a 2-isomorphism between the morphisms.

Example C.33. Homotopy equivalent topological spaces have isomorphic fundamental groupoids: a homotopy equivalence $f \colon X \to Y$ determines an isomorphism $\pi(f)_\bullet \colon \pi(X)_\bullet \to \pi(Y)_\bullet$.

Exercise C.36. Let X be a path connected topological space and $x \in X$ a base point. Let $G = \pi_1(X, x)$ be the fundamental group of X. Then the fundamental groupoid $\pi(X)_\bullet$ is isomorphic to $BG_\bullet$.

Exercise C.37. Prove that every transitive groupoid is isomorphic to a groupoid of the form $BG_\bullet$, for a group G. Every groupoid is isomorphic to a disjoint union $\coprod BG(i)_\bullet$, for some groups $G(i)$.

Exercise C.38. Let $X_\bullet$ be an equivalence relation, and let $Y = X_0/\cong$ be the set of equivalence classes. (a) Show that the canonical map $X_\bullet \to Y$ is an isomorphism of groupoids. In particular, if a group G acts freely on a set X, the transformation groupoid $X \rtimes G$ is isomorphic to the set of orbits. (b) Show that if Z is any set, an isomorphism $X_\bullet \to Z$ determines a bijection between $Y = X_0/\cong$ and Z.

Exercise C.39. If $X_\bullet$ and $Y_\bullet$ are isomorphic groupoids, show that $X_\bullet$ is rigid (resp. transitive) if and only if $Y_\bullet$ is rigid (resp. transitive).

Exercise C.40. A groupoid is rigid if and only if it is isomorphic to a set.

Exercise C.41. If $\phi_\bullet \colon X_\bullet \to Y_\bullet$ and $\psi_\bullet \colon Y_\bullet \to Z_\bullet$ are isomorphisms, then the composition $\psi_\bullet \circ \phi_\bullet \colon X_\bullet \to Z_\bullet$ is an isomorphism.

Exercise C.42. Suppose a set X has a left action of a group G and a right action of a group H, and these actions commute. Show that, if both actions are free, then the groupoids $G \ltimes (X/H)$ and $(G\backslash X) \rtimes H$ are isomorphic. For example, if H is a subgroup of a group G, then the groupoid $BH_\bullet$ is isomorphic to $G \ltimes (G/H)$.

Proposition C.34. *A morphism of groupoids $\phi_\bullet \colon X_\bullet \to Y_\bullet$ is an isomorphism if and only if it satisfies the following two conditions:*

(1) *For every $x, x' \in X_0$ and $b \in Y_1$ with $s(b) = \phi_0(x)$ and $t(b) = \phi_0(x')$, there is a unique $a \in X_1$ with $s(a) = x$, $t(a) = x'$, and $\phi_1(a) = b$. That is, the diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{(s,t)} & X_0 \times X_0 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_0 \times \phi_0} \\ Y_1 & \xrightarrow{(s,t)} & Y_0 \times Y_0 \end{array}$$

is a cartesian diagram of sets;

(2) *For every $y \in Y_0$, there is an $x \in X_0$ and a $b \in Y_1$ with $\phi_0(x) = s(b)$ and $t(b) = y$. That is, the map*

$$X_0 \;{}_{\phi_0}{\times}_{Y_0, s}\, Y_1 \to Y_0,$$

taking (x, b) to $t(b)$ is surjective. Equivalently, the induced map $X_0/{\cong} \to Y_0/{\cong}$ is surjective.

In the language of categories, the first condition says exactly that the functor $\phi_\bullet$ is faithful and full, and the second condition says that it is essentially surjective. A morphism of groupoids satisfying the first condition is said to be **injective**, and one satisfying the second will be called **surjective**.

The proof is largely left as an exercise, as it is the same as the corresponding result in category theory (Apendix B, §1). We remark only that the essential step in constructing a morphism $\psi_\bullet \colon Y_\bullet \to X_\bullet$ back is to *choose*, for each $y \in Y_0$, an $x_y \in X_0$ *and* a $b_y \in Y_1$ with $s(b_y) = \phi_0(x_y)$ and $t(b_y) = y$. Then set $\psi_0(y) = x_y$, and, for c in Y_1, set $\psi_1(c)$ to be the arrow from $\psi_0(s(c))$ to $\psi_0(t(c))$ such that $\phi_1(\psi_1(c)) = b_{s(c)} \cdot c \cdot {b_{t(c)}}^{-1}$.

Note that the second condition is automatic whenever ϕ_0 is surjective. In this case one need only choose x_y in X_0 with $\phi_0(x_y) = y$, and then one can take $b_y = e(y)$.

Remark C.35. The choices in this proof are important, not so much to point out the necessary use of an axiom of choice, but because they show that the inverse of an isomorphism may be far from canonical. This has serious consequences when the groupoids have a geometric structure on them. Set theoretic surjections have sections (by the axiom of choice). But geometric surjections, even nice ones like projections of fiber bundles, do not generally have sections. In particular, the classification of geometric groupoids is not as simple as it is for set-theoretic groupoids:

Corollary C.36. *Every groupoid is isomorphic to a family of groups as in Example C.9.*

Exercise C.43. A morphism $\phi_\bullet \colon X_\bullet \to Y_\bullet$ is an isomorphism if and only if the induced map $X_0/\cong \to Y_0/\cong$ is bijective and the induced maps $\mathrm{Aut}(x) \to \mathrm{Aut}(\phi_0(x))$ are isomorphisms for all $x \in X_0$.

Exercise C.44. If $X_\bullet$ and $Y_\bullet$ are isomorphic groupoids, show that $X_\bullet$ is finite (resp. tame) if and only if $Y_\bullet$ is finite (resp. tame), in which case they have the same mass.

Exercise C.45. A groupoid is rigid if and only if it is isomorphic to a set.

Exercise C.46. A banal groupoid is isomorphic to a point $pt \rightrightarrows pt$.

Exercise C.47. Suppose a set X has a left action of a group G and a right action of a group H, and these actions commute. Show that the canonical morphisms from the double transformation groupoid $G \ltimes X \rtimes H$ to $G \ltimes (X/H)$ (resp. $(G\backslash X) \rtimes H$) is an isomorphism if and only if the action of H (resp. G) on X is free. Deduce the result of Exercise C.42.

Exercise C.48. Construct a groupoid $X_\bullet$ from a set Z as in Exercise C.5. Let G be the group of bijections of Z with itself. There is a canonical surjective morphism from $G \ltimes X_0$ to $X_\bullet$, taking (σ, A) to $(A, \sigma(A), \sigma|_A)$. For which Z is this an isomorphism?

Exercise C.49. Any linear map $L \colon V \to W$ of vector spaces (or abelian groups) determines an action of V on W by translation: $v \cdot w = L(v) + w$, and so we have the transformation groupoid $V \ltimes W$. If $L' \colon V' \to W'$ is another, a pair of linear maps $\phi_V \colon V \to V'$, $\phi_W \colon W \to W'$, such that $L' \circ \phi_V = \phi_W \circ L$ determines a homomorphism $\phi_\bullet \colon V \ltimes W \to V' \ltimes W'$. (a) Show that $\phi_\bullet$ is an isomorphism if and only if the induced maps $\mathrm{Ker}(L) \to \mathrm{Ker}(L')$ and $\mathrm{Coker}(L) \to \mathrm{Coker}(L')$ are isomorphisms. (b) Show that $V \ltimes W$ is isomorphic to the groupoid $\mathrm{Ker}(L) \ltimes \mathrm{Coker}(L)$ (with the trivial action), which is isomorphic to the product of $B(\mathrm{Ker}(L))_\bullet$ and the set $\mathrm{Coker}(L)$.

Exercise C.50. If a group G acts on the right on groupoids $X_\bullet$ and $Y_\bullet$, a morphism $\phi_\bullet \colon X_\bullet \to Y_\bullet$ is G-**equivariant** if ϕ_0 and ϕ_1 are G-equivariant. There is then an induced morphism $X_\bullet \rtimes G \to Y_\bullet \rtimes G$. Show that, if $\phi_\bullet$ is an isomorphism, then this induced morphism is also an isomorphism.

5. Fibered products

Let

$$\begin{array}{ccc} & & Y_\bullet \\ & & \big\downarrow{\scriptstyle \psi_\bullet} \\ X_\bullet & \xrightarrow{\phi_\bullet} & Z_\bullet \end{array}$$

be a diagram of groupoids. We shall construct

(i) a groupoid $W_\bullet$;
(ii) two morphisms of groupoids $p_\bullet\colon W_\bullet \to X_\bullet$ and $q_\bullet\colon W_\bullet \to Y_\bullet$;
(iii) a 2-isomorphism θ between the compositions $W_\bullet \to X_\bullet \to Z_\bullet$ and $W_\bullet \to Y_\bullet \to Z_\bullet$.

The data $(W_\bullet, p_\bullet, q_\bullet, \theta)$ will be called the **fibered product** of $X_\bullet$ and $Y_\bullet$ over $Z_\bullet$, notation $W_\bullet = X_\bullet \times_{Z_\bullet} Y_\bullet$.

(1)
$$\begin{array}{ccc} W_\bullet & \xrightarrow{q_\bullet} & Y_\bullet \\ {\scriptstyle p_\bullet}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle \psi_\bullet} \\ X_\bullet & \xrightarrow[\phi_\bullet]{} & Z_\bullet \end{array}$$

The objects of $W_\bullet$ are triples (x, y, c), where x and y are objects of $X_\bullet$ and $Z_\bullet$, respectively, and c is a morphism in $Z_\bullet$, between $\phi_0(x)$ and $\psi_0(y)$:

$$\phi_0(x) \xrightarrow{c} \psi_0(y)$$

Given two such objects (x, y, c) and (x', y', c') define a morphism from (x, y, c) to (x', y', c') to be a pair (a, b), $x \xrightarrow{a} x'$, $y \xrightarrow{b} y'$, such that

$$\begin{array}{ccc} \phi_0(x) & \xrightarrow{\phi_1(a)} & \phi_0(x') \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ \psi_0(y) & \xrightarrow{\psi_1(b)} & \psi_0(y') \end{array}$$

commutes in $Z_\bullet$. Composition in $W_\bullet$ is induced from composition in $X_\bullet$ and $Y_\bullet$.

The two projections $p_\bullet$ and $q_\bullet$ are defined by projecting onto the first and second components, respectively (both on objects and morphisms).

To define $\theta\colon \phi_\bullet \circ p_\bullet \to \psi_\bullet \circ q_\bullet$, take $\theta\colon X_0 \to Y_1$ to be the map $(x, y, c) \mapsto c$.

This fibered product satisfies a **universal mapping property**: given a groupoid $V_\bullet$ and two morphisms $f_\bullet\colon V_\bullet \to X_\bullet$ and $g_\bullet\colon V_\bullet \to Y_\bullet$, together with a 2-isomorphism $\tau\colon \phi_c om \circ f_\bullet \Rightarrow \psi_c om \circ g_\bullet$, there is a unique morphism $h_\bullet = (f_\bullet, g_\bullet)\colon V_\bullet \to X_\bullet \times_{Z_\bullet} Y_\bullet$ such that $f_\bullet = p_\bullet \circ h_\bullet$, $g_\bullet = q_\bullet \circ h_\bullet$, and τ is determined from θ by $\tau = \theta * 1_{h_\bullet}$. In fact, $h_\bullet$ is defined by $h_0(v) = (f_0(v), g_0(v), \tau(v))$ for $v \in V_0$, and $h_1(d) = (f_1(d), g_1(d))$ for $d \in V_1$.

A **2-commutative diagram**

$$\begin{array}{ccc} V_\bullet & \xrightarrow{g_\bullet} & Y_\bullet \\ {\scriptstyle f_\bullet}\downarrow & \overset{\tau}{\nearrow} & \downarrow{\scriptstyle \psi_\bullet} \\ X_\bullet & \xrightarrow[\phi_\bullet]{} & Z_\bullet \end{array}$$

means that a 2-isomorphism τ from $\phi_\bullet \circ f_\bullet$ to $\psi_\bullet \circ g_\bullet$ is specified. It **strictly commutes** in case $\phi_\bullet \circ f_\bullet = \psi_\bullet \circ g_\bullet$. In this case the 2-isomorphism is taken to be $\epsilon\colon V_0 \to Z_1$ given by $\epsilon = e \circ \phi_o \circ f_0 = e \circ \psi_o \circ g_0$.

Exercise C.51. Show that a 2-commutative diagram strictly commutes exactly when the 2-isomorphism $\theta\colon V_0 \to Z_1$ factors through Z_0, i.e., $\theta = e \circ \theta_0$ for some map $\theta_0\colon V_0 \to Z_0$.

The diagram is called **2-cartesian** if it is 2-commutative and the induced mapping $(f_\bullet, g_\bullet)\colon V_\bullet \to X_\bullet \times_{Z_\bullet} Y_\bullet$ is an isomorphism. Such a $V_\bullet$ will not satisfy the same universal property as the fibered product we have constructed; but it does satisfy a universal property in an appropriate 2-categorical sense (see Exercise C.56). The universal property just described is easier to use in practise.

The diagram is called **strictly 2-cartesian** if the induced mapping $(f_\bullet, g_\bullet)\colon V_\bullet \to X_\bullet \times_{Z_\bullet} Y_\bullet$ is a strict isomorphism.

Example C.37. Let X be a right G-set and $X \rtimes G$ the associated transformation groupoid. Consider the diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{\sigma} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & X \rtimes G, \end{array}$$

where σ is the action map and π is the canonical map. This diagram does not strictly commute, so we consider the 2-isomorphism $\eta\colon \pi \circ p_1 \to \pi \circ \sigma$ given by the identity map on $X \times G$. This gives a 2-commutative diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{\sigma} & X \\ {\scriptstyle p_1}\downarrow & \eta \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & X \rtimes G, \end{array} \tag{2}$$

and it is not difficult to see that the corresponding map from the set $X \times G$ to the fibered product $X \times_{X \rtimes G} X$ is a *strict* isomorphism. Thus $X \rtimes G$ can be considered to be a quotient of X by G, but a much better quotient than the set-theoretic quotient, because the set-theoretic quotient does not make the corresponding Diagram (2) a cartesian diagram of sets (or groupoids).

Remark C.38. Diagram (2) also has a 'dual' property, which expresses the fact that $X \rtimes G$ is a quotient of X by the action of G. This property is that for every set S and every morphism $f\colon X \to S$, such that $f \circ p_1 = f \circ \sigma$ there exists a unique morphism $\overline{f}\colon X \rtimes G \to S$ such that $\overline{f} \circ \pi = f$:

$$\begin{array}{ccccc} X \times G & \overset{\sigma}{\underset{p_1}{\rightrightarrows}} & X & \xrightarrow{\pi} & X \rtimes G \\ & & & {\scriptstyle f}\searrow & \vdots\, {\scriptstyle \overline{f}} \\ & & & & S \end{array}$$

We refer to this property as the *cocartesian* property of Diagram (2). There is also a more complicated version of this property for an arbitrary groupoid in place of the set S, for which we refer to Exercise C.53.

EXERCISE C.52. Generalize the previous example by replacing the transformation groupoid $X \rtimes G$ by an arbitrary groupoid $X_\bullet$. In other words, construct a 2-cartesian diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{t} & X_0 \\ {\scriptstyle s}\downarrow & \overset{\eta}{\nearrow} & \downarrow{\scriptstyle \pi} \\ X_0 & \xrightarrow{\pi} & X_\bullet. \end{array}$$

Show in fact that X_1 is strictly isomorphic to the fibered product $X_0 \times_{X_\bullet} X_0$. This diagram also has a cocartesian property with respect to maps into sets S.

EXERCISE C.53. Show that for any groupoid $X_\bullet$, the morphism $\pi\colon X_0 \to X_\bullet$ makes $X_\bullet$ a 2-quotient of X_0 in the 2-category (Gpd) (in the sense of Definition B.17).

EXERCISE C.54. If $X_\bullet = X$ and $Y_\bullet = Y$ are sets, so $\phi_\bullet$ and $\psi_\bullet$ are given by maps $f\colon X \to Z_0$ and $g\colon Y \to Z_0$, then the fibered product $X \times_{Z_\bullet} Y$ is strictly isomorphic to the set

$$W = \{\, (x, y, c) \in X \times Y \times Z_1 \mid s(c) = f(x) \text{ and } t(c) = g(y) \,\}.$$

In the preceding exercise, if $Y = X$ and $g = f$, one gets a 2-cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{t} & X \\ {\scriptstyle s}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Z_\bullet, \end{array}$$

with $W = \{(y_1, y_2, a) \in Y \times Y \times Z_1 \mid f(y_1) \xrightarrow{a} f(y_2)\}$, and $\theta\colon W \to Z_1$ is the third projection.

EXAMPLE C.39. Let $W_\bullet$ be the fibered product $(X \rtimes G) \times_{BG_\bullet} pt$. From the construction of the fibered product we can identify W_0 with $X \times G$ and W_1 with $X \times G \times G$, with $s(x, g, h) = (x, gh)$, $t(x, g, h) = (xg, h)$, and $(x, g, h) {\cdot} (xg, g', h') = (x, gg', h')$.

EXERCISE C.55. Show that the canonical morphism $\alpha_\bullet\colon X \to W_\bullet$, defined by $\alpha_0(x) = (x, e)$ and $\alpha_1(x) = (x, e, e)$, satisfies the conditions of Proposition C.34, so $\alpha_\bullet$ is an isomorphism. Thus the diagram

$$\begin{array}{ccc} X & \longrightarrow & pt \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X \rtimes G & \longrightarrow & BG_\bullet \end{array}$$

is 2-cartesian. Construct a morphism $\beta_\bullet\colon W_\bullet \to X$ by the formulas $\beta_0(x, g) = xg$ and $\beta_1(x, g, h) = xgh$. Verify that $\beta_\bullet \circ \alpha_\bullet = 1_X$, and construct a 2-isomorphism from $\alpha_\bullet \circ \beta_\bullet$ to $1_{W_\bullet}$.

Note that $X \to X \rtimes G$ is the "general" quotient by G. Thus we see that every quotient by G is a pullback from the quotient of pt by G (which is $BG_\bullet$). This justifies calling $pt \to BG_\bullet$ the *universal* quotient by G.

Exercise C.56. (*) Show that a 2-commutative diagram is 2-cartesian as defined here if and only if it is 2-cartesian in the the 2-category of groupoids, i.e., it satisfies the universal property of Appendix B, Definition B.17.

Note how this universal mapping property characterizes the fibered product $W_\bullet$ up to an isomorphism which is unique up to a unique 2-isomorphism. This is the natural analogue in a 2-category of the usual 'unique up to unique isomorphism' in an ordinary category.

5.1. Square morphisms.

Definition C.40. A morphism of groupoids $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is called **square** if the diagrams

$$\begin{array}{ccc} X_1 & \xrightarrow{s} & X_0 \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_0} \\ Y_1 & \xrightarrow{s} & Y_0 \end{array} \qquad\qquad \begin{array}{ccc} X_1 & \xrightarrow{t} & X_0 \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_0} \\ Y_1 & \xrightarrow{t} & Y_0 \end{array}$$

are cartesian diagrams of sets. Since s and t are obtained from each other by the involution i, it suffices to verify that one of these diagrams is cartesian.

Exercise C.57. The morphism $X \rtimes G \to BG_\bullet$ of Example C.23 is square.

Exercise C.58. If $X_\bullet$ is a groupoid, then any square morphism $X_\bullet \to BG_\bullet$ makes $X_\bullet$ strictly isomorphic to a transformation groupoid associated to an action of G on X_0.

5.2. Restrictions and Pullbacks.

Definition C.41. Let $X_\bullet$ be a groupoid, Y_0 a set and $\phi_0\colon Y_0 \to X_0$ a map. Define Y_1 to be the fibered product (of sets)

$$\begin{array}{ccc} Y_1 & \xrightarrow{(s,t)} & Y_0 \times Y_0 \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_0\times\phi_0} \\ X_1 & \xrightarrow{(s,t)} & X_0 \times X_0. \end{array}$$

So an element of Y_1 is a triple $(y, y', a) \in Y_0 \times Y_0 \times X_1$ with $\phi_0(y) \xrightarrow{a} \phi_0(y')$. Define the structure of a groupoid on $Y_\bullet$ by the rule

$$(y, y', a)\cdot(y', y'', b) = (y, y'', a\cdot b).$$

We get an induced morphism of groupoids $\phi_\bullet\colon Y_\bullet \to X_\bullet$, defined by $\phi_1(y, y', a) = a$.

The groupoid $Y_\bullet$ is called the **restriction** of $X_\bullet$ via $Y_0 \to X_0$; following [**50**], it may be denoted $X_\bullet|_{Y_0}$.[4]

[4] [The word "pullback" and the notation $\phi_0^*(X_\bullet)$ might seem more appropriate, since "restriction" connotes some kind of subobject, but the word pullback is used for another concept.]

Note that by construction, $Y_\bullet \to X_\bullet$ is injective (full and faithful). It is an isomorphism exactly when the image of the map $Y_0 \to X_0$ intersects all isomorphism classes of $X_\bullet$, by Proposition C.34.

Example C.42. Let X be a right G-set and $U \subset X$ a subset. The restriction of $X \rtimes G$ to U is not a transformation groupoid unless U is G-invariant. Thus we see that very natural constructions can lead out of the world of group actions.

Example C.43. If $\pi(X)_\bullet$ is the fundamental group of a topological space X, and A is a subset of X, then the restriction of $\pi(X)_\bullet$ to A is the groupoid $\pi(X, A)_\bullet$.

Exercise C.59. Show that any morphism : $X_\bullet \to Y_\bullet$ factors canonically into $X_\bullet \to Y'_\bullet \to Y_\bullet$, with $X_0 \to Y'_0$ injective, and $Y'_\bullet \to Y_\bullet$ an isomorphism.

Definition C.44. Let $X_\bullet$ be a groupoid, and $f\colon X_0 \to Z$ a map to a set Z such that $f \circ s = f \circ t$. For any map $Z' \to Z$, construct a **pullback** groupoid $X'_\bullet$ by setting $X'_0 = X_0 \times_Z Z'$, $X'_1 = X_1 \times_Z Z'$, with s' and t' induced by s and t, as is m' from m, by means of the isomorphism $X'_1 \,{}_{t'}\!\times_{X'_0,s'} X'_1 \cong (X_1 \,{}_t\!\times_{X_0,s} X_1) \times_Z Z'$.

Exercise C.60. Verify that $X'_\bullet$ is a groupoid. Show that the induced morphism $X'_\bullet \to X_\bullet$ is square.

5.3. Representable and gerbe-like morphisms.

Definition C.45. A morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ of groupoids is called **representable** if the induced mapping

$$(s, t, \phi_1)\colon X_1 \longrightarrow (X_0 \times X_0) \times_{Y_0 \times Y_0} Y_1$$

is injective; that is, $\phi_\bullet$ is faithful as a functor between categories. The morphism is said to be **gerbe-like** if this map (s, t, ϕ_1) is surjective, and the induced map $X_0/\cong \to Y_0/\cong$ is surjective; that is, $\phi_\bullet$ is a full and essentially surjective functor. So a representable and gerbe-like morphism is an isomorphism.

For any groupoid $X_\bullet$, the canonical morphism $X_0 \to X_\bullet$ is representable (but not usually injective). If $X'_\bullet$ is a pullback of $X_\bullet$, as defined in the last section, the map $X'_\bullet \to X_\bullet$ is representable.

The canonical morphism from $X_\bullet$ to $X_0/\cong$ is gerbe-like. Any surjective homomorphism $G \to H$ of groups determines a gerbe-like homomorphism $BG_\bullet \to BH_\bullet$.

Exercise C.61. Let $\phi_\bullet\colon X_\bullet \to Y_\bullet$ be a morphism of groupoids. The following are equivalent:

(i) $\phi_\bullet$ is representable;
(ii) For any set T and morphism $T \to Y_\bullet$, the fibered product $X_\bullet \times_{Y_\bullet} T$ is rigid;
(iii) For any rigid groupoid $T_\bullet$ and morphism $T_\bullet \to Y_\bullet$, the fibered product $X_\bullet \times_{Y_\bullet} T_\bullet$ is rigid;
(iv) For any 2-cartesian diagram

$$\begin{array}{ccc} S_\bullet & \longrightarrow & T_\bullet \\ \big\downarrow & \overset{\alpha}{\nearrow} & \big\downarrow \\ X_\bullet & \longrightarrow & Y_\bullet \end{array}$$

with $T_\bullet$ rigid, $S_\bullet$ is also rigid.

(v) For any set T and morphism $T \to Y_\bullet$, there is a set S and a 2-cartesian diagram

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & \overset{\alpha}{\nearrow} & \downarrow \\ X_\bullet & \longrightarrow & Y_\bullet . \end{array}$$

Exercise C.62. Show that any morphism $X_\bullet \to Y_\bullet$ factors canonically into a gerbe-like morphism $X_\bullet \to Z_\bullet$ followed by a representable morphism $Z_\bullet \to Y_\bullet$.

Exercise C.63. For a morphism $\phi_\bullet \colon X_\bullet \to Y_\bullet$ of groupoids, show that the following are equivalent:

(i) $\phi_\bullet$ is gerbe-like;
(ii) For any morphism $pt \to Y_\bullet$ (given by $y \in Y_0$), the fibered product $X_\bullet \times_{Y_\bullet} pt$ is non-empty and transitive.
(iii) For any morphism $\psi_\bullet \colon\ pt \to Y_\bullet$, there is a group G and a 2-cartesian diagram

$$\begin{array}{ccc} BG_\bullet & \longrightarrow & pt \\ \downarrow & \overset{\alpha}{\nearrow} & \downarrow \psi_\bullet \\ X_\bullet & \xrightarrow{\phi_\bullet} & Y_\bullet \end{array}$$

6. Simplicial constructions

We fix a groupoid $X_\bullet$ and explain several constructions of new groupoids out of $X_\bullet$. For any integer $n \geq 1$, denote by X_n the set of n composable morphisms in $X_\bullet$, i.e.,

$$X_n \;=\; \{\, (a_1, \ldots, a_n) \in (X_1)^n \mid t(a_i) = s(a_{i+1}) \text{ for } 1 \leq i < n \,\} :$$

$$* \xrightarrow{a_1} * \xrightarrow{a_2} \cdots \xrightarrow{a_n} *$$

6.1. Groupoid of diagrams. Let $X_\bullet$ be a groupoid. Define a new groupoid $X_\bullet\{n\}$, for $n \geq 1$ as follows. An object of $X_\bullet\{n\}$ is an n-tuple of composable arrows in $X_\bullet$, i.e., an element of X_n. A morphism in $X_\bullet\{n\}$ from $(a_1, \ldots, a_n) \in X_n$ to $(b_1, \ldots, b_n) \in X_n$ is a commutative diagram in $X_\bullet$

$$\begin{array}{ccccccc} * & \xrightarrow{a_1} & * & \xrightarrow{a_2} & \cdots & \xrightarrow{a_n} & * \\ \phi_0 \downarrow & & \phi_1 \downarrow & & & & \downarrow \phi_n \\ * & \xrightarrow{b_1} & * & \xrightarrow{b_2} & \cdots & \xrightarrow{b_n} & * \end{array}$$

i.e., an $(n+1)$-tuple $(\phi_0, \ldots, \phi_n)$ of arrows in $X_\bullet$ such that $\phi_{i-1}{\cdot}b_i = a_i{\cdot}\phi_i$, for all $i = 1, \ldots, n$.

Composition in $X_\bullet\{n\}$ is defined by composing vertically:

$$(\phi_0, \ldots, \phi_n){\cdot}(\psi_0, \ldots, \psi_n) = (\phi_0{\cdot}\psi_0, \ldots, \phi_n{\cdot}\psi_n).$$

We call the groupoid $X_\bullet\{n\}$ the **groupoid of n-diagrams** of $X_\bullet$.

EXERCISE C.64. Construct a strict isomorphism between $X_\bullet\{1\}$ and the restriction of $X_\bullet$ by the map $s\colon X_1 \to X_0$. More generally, construct a strict isomorphism between $X_\bullet\{n\}$ and the restriction of $X_\bullet$ by the map from X_n to X_0 that takes $(a_1,\dots,a_n)$ to $s(a_1)$. Conclude that all of the groupoids $X_\bullet\{n\}$ are isomorphic to $X_\bullet$.

EXERCISE C.65. Define a groupoid $V_\bullet^{(n)}$ with $V_0^{(n)} = X_n$, $V_1^{(n)} = X_{2n+1}$, $s(a_1,\dots,a_n,c,b_1,\dots,b_n) = (a_n^{-1},\dots,a_1^{-1})$, and $t(a_1,\dots,a_n,c,b_1,\dots,b_n) = (b_1,\dots,b_n)$. Construct a strict isomorphism between $V_\bullet^{(n)}$ and $X_\bullet\{n\}$. Deduce that $X_\bullet\{n\}\{1\}$ is strictly isomorphic to $X_\bullet\{2n+1\}$. Prove more generally that $X_\bullet\{n\}\{m\}$ is strictly isomorphic to $X_\bullet\{(n+1)(m+1)-1\}$.

DEFINITION C.46. Define the **shift of $X_\bullet$ by n** to be the subgroupoid $X_\bullet[n]$ of $X_\bullet\{n\}$ defined by

$$\big(X_\bullet[n]\big)_0 = \big(X_\bullet\{n\}\big)_0 = X_n$$
$$\big(X_\bullet[n]\big)_1 = \{(\phi_0,\dots,\phi_n) \in \big(X_\bullet\{n\}\big)_1 \mid \phi_1,\dots,\phi_n \text{ are identity morphisms}\}.$$

EXERCISE C.66. (1) Define a groupoid $W_\bullet^{(n)}$ by $W_0^{(n)} = X_n$, $W_1^{(n)} = X_{n+1}$, with $s(a_1,\dots,a_{n+1}) = (a_1{\cdot}a_2,a_3,\dots,a_{n+1})$, $t(a_1,\dots,a_{n+1}) = (a_2,a_3,\dots,a_{n+1})$, and

$$(a_1,\dots,a_{n+1}){\cdot}(b_1,\dots,b_{n+1}) = (a_1{\cdot}b_1,b_2,\dots,b_{n+1}).$$

(2) Construct a strict isomorphism between $W_\bullet^{(n)}$ and the cross product groupoid $X_n \times_{X_{n-1}} X_n \rightrightarrows X_n$, constructed from the morphism $X_n \to X_{n-1}$ that maps $(a_1,\dots,a_n)$ to $(a_2,\dots,a_n)$. (3) Show that $W_\bullet^{(n)}$ is strictly isomorphic to $X_\bullet\{n\}$.

EXERCISE C.67. Define a morphism $X_\bullet[n+1] \to X_\bullet[n]$ by leaving out the last component. Prove that this morphism is square.

EXERCISE C.68. (*) For $0 \le k \le n$, and $n \ge 2$, define $d_k\colon X_n \to X_{n-1}$ by the formulas $d_0(a_1,\dots,a_n) = (a_2,\dots,a_n)$, $d_k(a_1,\dots,a_n) = (a_1,\dots,a_k{\cdot}a_{k+1},\dots,a_n)$ for $0 < k < n$, and $d_n(a_1,\dots,a_n) = (a_1,\dots,a_{n-1})$. For any $1 \le k \le n$, construct a groupoid $U_\bullet = X_\bullet(n,k)$ with $U_0 = X_{n-1}$, $U_1 = X_n$, $s = d_k$, $t = d_{k-1}$, and

$$(a_1,\dots,a_n){\cdot}(b_1,\dots,b_n) = (a_1,\dots,a_{k-1},a_k{\cdot}b_k,b_{k+1},\dots,b_n).$$

(1) Show that $X_\bullet(n,k)$ is strictly isomorphic to $X_\bullet(n,l)$ for any $1 \le k,l \le n$. (2) The formulas $\phi_1(a_1,\dots,a_n) = a_k$ and $\phi_0(a_1,\dots,a_{n-1}) = s(a_k)$ determine a morphism $\phi_\bullet\colon U_\bullet \to X_\bullet$. Show that this morphism is faithful and essentially surjective, but not usually full.

6.2. Simplicial sets.

DEFINITION C.47. A **simplicial set** X_* specifies a set X_n of **n-simplices** for each nonnegative integer n, together with **face** maps $d_i\colon X_n \to X_{n-1}$ for $0 \le i \le n$, and **degeneracy** maps $s_i\colon X_n \to X_{n+1}$ for $0 \le i \le n$, satisfying the following identities:

(a) $d_i d_j = d_{j-1} d_i$ for $i < j$;
(b) $s_i s_j = s_{j+1} s_i$ for $i \le j$;

(c) $d_i s_j = \begin{cases} s_{j-1}d_i & \text{for} \quad i < j; \\ \text{id} & \text{for} \quad i = j, j+1; \\ s_j d_{i-1} & \text{for} \quad i > j+1. \end{cases}$

A groupoid $X_\bullet$ determines a simplicial set X_*, called the **simplicial set of the groupoid**, whose set of n-simplices is the set X_n of composable morphisms $(a_1, \dots, a_n)$ in $X_\bullet$, with X_0 the objects of $X_\bullet$. For $n = 1$, $d_0 = t$ and $d_1 = s$ are the two maps from X_1 to X_0, and $s_0 = e$ is the map from X_0 to X_1. The general maps are defined by:

$$d_i(a_1, \dots, a_n) = \begin{cases} (a_2, \dots, a_n) & \text{if} \quad i = 0; \\ (a_1, \dots, a_i{\cdot}a_{i+1}, \dots, a_n) & \text{if} \quad 0 < i < n; \\ (a_1, \dots, a_{n-1}) & \text{if } i = n. \end{cases}$$

and

$$s_i(a_1, \dots, a_n) = \begin{cases} (1_{s(a_1)}, a_1, \dots, a_n) & \text{if} \quad i = 0; \\ (a_1, \dots, a_i, 1_{t(a_i)=s(a_{i+1})}, a_{i+1}, \dots, a_n) & \text{if} \quad 0 < i < n; \\ (a_1, \dots, a_n, 1_{t(a_n)}) & \text{if} \quad i = n. \end{cases}$$

Exercise C.69. Verify (a), (b), and (c), so X_* is a simplicial set.

A **morphism** $\phi_*\colon X_* \to Y_*$ of simplicial sets is given by a mapping $\phi_n\colon X_n \to Y_n$ for each $n \geq 0$, commuting with the face and degeneracy operators. A morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ of groupoids determines a morphism $\phi_*\colon X_* \to Y_*$ of their simplicial sets, where ϕ_0 and ϕ_1 are the given maps, and $\phi_n(a_1, \dots, a_n) = (\phi_1(a_1), \dots, \phi_1(a_n))$ for $n \geq 1$. If ϕ_* and ψ_* are morphisms from X_* to Y_*, a **homotopy** h from ϕ_* to ψ_* is given by a collection of maps $h_i\colon X_n \to Y_{n+1}$ for all $0 \leq i \leq n$, satisfying:

(a) $d_0 h_0 = \phi_n$ and $d_{n+1} h_n = \psi_n$;

(b) $d_i h_j = \begin{cases} h_{j-1} d_i & \text{if} \quad i < j; \\ d_j h_{j-1} & \text{if} \quad i = j > 0; \\ h_j d_{i-1} & \text{if} \quad i = n. \end{cases}$

(c) $s_i h_j = \begin{cases} h_{j+1} s_i & \text{if} \quad i \leq j; \\ h_j s_{i-1} & \text{if} \quad i > j. \end{cases}$

Exercise C.70. If $\theta\colon X_0 \to Y_1$ gives a 2-isomorphism between morphisms $\phi_\bullet$ and $\psi_\bullet$ from a groupoid $X_\bullet$ to a groupoid $Y_\bullet$, show that the mappings $h_i\colon X_n \to Y_{n+1}$ defined by

$$h_i(a_1, \dots, a_n) = (\phi_1(a_1), \dots \phi_1(a_i), \theta(t(a_i)) = \theta(s(a_{i+1})), \psi_1(a_{i+1}), \dots, \psi_1(a_n))$$

defines a homotopy from ψ_* to ϕ_*.

Definition C.48. A simplicial set X_* satisfies the **Kan condition** if, for every $0 \leq k \leq n$ and sequence $\sigma_0, \dots, \sigma_{k-1}, \sigma_{k+1}, \dots, \sigma_n$ of n $(n-1)$-simplices satisfying $d_i(\sigma_j) = d_{j-1}(\sigma_i)$ for all $i < j$ and $i \neq k \neq j$, there is a σ in X_n with $d_i(\sigma) = \sigma_i$ for all $i \neq k$. This condition is the simplicial analogue of the fact that the union of n faces of an n-simplex is a retract of the simplex. The Kan condition implies that the condition of being homotopic is an equivalence relation. It also implies that the homotopy groups of the geometric realization of the simplicial set can be computed combinatorially. For these and other facts about simplicial sets we refer to [**60**] and [**67**].

Exercise C.71. Show that the simplicial set of a groupoid satisfies the Kan condition. [For $k = 0$, and $\sigma_1 = (b_1, \dots, b_{n-1})$ and $\sigma_2 = (c_1, \dots, c_{n-1})$, the other σ_i are determined, and one may take $\sigma = (c_1, c_1^{-1}{\bullet}b_1, b_2, \dots, b_{n-1})$. For $k = 1$, $\sigma_0 = (a_1, \dots, a_{n-1})$ and $\sigma_2 = (c_1, \dots, c_{n-1})$, take $\sigma = (c_1, a_1, a_2, \dots, a_{n-1})$. For $k > 1$, $\sigma_0 = (a_1, \dots, a_{n-1})$ and $\sigma_1 = (b_1, \dots, b_{n-1})$, take $\sigma = (b_1{\bullet}a_1^{-1}, a_1, a_2, \dots, a_{n-1})$.]

Definition C.49. The **standard n-simplex** $\Delta(n)$ is defined by

$$\Delta(n) = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1 \}.$$

regarded as a topological subspace of Euclidean space. For a simplicial set X_*, construct the topological space

$$X = \coprod_{n \geq 0} X_n \times \Delta(n).$$

Topologically, X is the disjoint union of copies of the standard n-simplex, with one for each n-simplex in X_*. Define the **geometric realization** $|X_*|$ of X_* to be the quotient space $X/\sim$ of X by the equivalence relation generated by all

$$(d_i(\sigma), (t_0, \dots, t_{n-1})) \sim (\sigma, (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

for $\sigma \in X_n$, $(t_0, \dots, t_{n-1}) \in \Delta(n-1)$, $0 \leq i \leq n$, and

$$(d_i(\sigma), (t_0, \dots, t_{n+1})) \sim (\sigma, (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1})$$

for $\sigma \in X_n$, $(t_0, \dots, t_{n+1}) \in \Delta(n+1)$, $0 \leq i \leq n$. An n-simplex σ in X_n is called **nondegenerate** if it does not have the form $s_i(\tau)$ for $\tau \in X_{n-1}$ and some $0 \leq i \leq n-1$. For each n-simplex σ there is a continuous mapping from $\Delta(n)$ to $|X_*|$ that takes $t \in \Delta(n)$ to the equivalence class of (σ, t). If σ is nondegenerate, this maps the interior of $\Delta(n)$ homeomorphically onto its image. The space $|X_*|$ is a CW-complex, with these images as its cells.

A morphism $\phi_* \colon X_* \to Y_*$ determines a continuous mapping $|\phi_*| \colon |X_*| \to |Y_*|$. Homotopic mappings of simplicial sets determine homotopic mappings between their geometric realizations.

Exercise C.72. Any topological space X determines a simplicial set $S_*(X)$, where $S_n(X)$ is the set of all continuous mappings σ from the standard n-simplex to X, with $(d_i\sigma)(t_0, \dots, t_{n-1}) = \sigma(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$ and $(s_i\sigma)(t_0, \dots, t_{n+1}) = \sigma(t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_n)$, for $\sigma \in S_n(X)$ and $0 \leq i \leq n$. A continuous mapping $f \colon X \to Y$ determines a mapping $S_*(f) \colon S_*(X) \to S_*(Y)$ of simplicial sets, so we have a functor from (Top) to the category (Sss) of simplicial sets. This functor is a right adjoint to the geometric realization functor from (Sss) to (Top): if X_* is a simplicial set and Y is a topological space, there is a canonical bijection

$$\operatorname{Hom}(X_*, S_*(Y)) \longleftrightarrow \operatorname{Hom}(|X_*|, Y).$$

In fact, 2-isomorphisms of simplicial sets correspond to homotopies between spaces, so one has a strict isomorphism of categories $\mathrm{HOM}(X_*, S_*(Y)) \cong \mathrm{HOM}(|X_*|, Y)$. [See [**67**], §16.]

[What is the relation between a groupoid $X_\bullet$ and the (relative) fundamental groupoid $\pi(|X_*|, X_0)_\bullet$? Should we define product of simplicial sets? Should we point out that a simplicial set is the same thing as a contravariant functor from the category $\mathcal{V}$ to (Set), where $\mathcal{V}$ is the category with one object $\{0, \ldots, n\}$ for each nonnegative integer, and with morphisms nondecreasing mappings between such sets. And/or say that both definitions make sense for (Set) replaced by any category? Define the simplicial set I_* and state that a homotopy is the same as $X_* \times I_* \to Y_*$ ([**67**], §6)?

There is a fancier 2-categorical notion in Barbara's chapter on group actions on stacks that could appear in this appendix? What else is needed in the text?]

Answers to Exercises

C.2. $e(x)$ is determined by the category properties (i)–(iv), as the identity of the monoid $\{a \in X_1 \mid s(a) = x, t(a) = x\}$. If $i(f) \cdot f = et(f)$ and $f \cdot i(f) = es(f)$, then $i(f) = i(f) \cdot (f \cdot i'(f)) = (i(f) \cdot f) \cdot i'(f) = i'(f)$. The proofs of identities (vii)–(ix) are similar to those in group theory.

C.6. The associativity is proved just as in the case of free groups.

C.7. The unity takes value 1 on $e(X_0)$ and 0 on the complement.

C.11. $G \ltimes X_\bullet$ is $G \times X_1 \rightrightarrows X_0$, with $s(g, a) = s(a)$, $t(g, a) = t(ga)$, and $(g, a) \cdot (g', a') = (g'g, a \cdot g^{-1}a')$.

C.12. Each is (canonically) strictly isomorphic to a $G \ltimes X_\bullet \rtimes H$, which is the groupoid $G \times X_1 \times H \rightrightarrows X_0$, with $s(g, a, h) = s(a)$, $t(g, a, h) = t(gah)$, and $(g, a, h) \cdot (g', a', h') = (g'g, a \cdot g^{-1}a'h^{-1}, hh')$.

C.13. The data $s, t \colon X_1 \to X_0$ determine a directed graph Γ. Form X by adjoining a disk for each identity map 1_x, $x \in X_0$, and a triangle for each $(a, b) \in X_2$: [pictures of disks bounding an arrow at x and a triangle with sides a, b, and a $\cdot$ b should be drawn here] Take A to be the set X_0 of vertices. See Section 6.2 for more general constructions.

C.17. If ϕ_a is defined by a and $\phi_{a'}$ is defined by a', then $\phi_{a'}(g) = z^{-1}\phi_a(g)z$, with $z = a^{-1} \cdot a'$.

C.20. The mass is $\frac{1}{(q+1)(q^3-1)}$.

C.21. This is the restriction of $X_\bullet$ from the canonical map from E_0 to X_0, cf. C.41.

C.29. To verify C.25, look at the map $(s, t) \mapsto H(a(s), t)$, which has $s \mapsto f(a(s))$ on the bottom, $s \mapsto g(a(s))$ on the top, $t \mapsto H(a(0), t)$ on the left side, and $t \mapsto H(a(1), t)$ on the right.

C.32. The only possible 2-isomorphism from $f_\bullet$ to $g_\bullet$ is given by $\theta(x) = (f_0(x), g_0(x)) \in Y_1 \subset Y_0 \times Y_0$.

C.35. A morphism from $X_\bullet \times I_\bullet$ to $Y_\bullet$ is given by a pair of maps $f_0, f_1 \colon X_0 \to Y_0$, and four maps $f_{00}, f_{01}, f_{10}, f_{11} \colon X_1 \to Y_1$, satisfying some identities. The bijection is given by

$$\phi_0 = f_0,\ \psi_0 = f_1,\ \phi_1 = f_{00},\ \psi_1 = f_{11},\ \theta = f_{01} \circ e,\ f_{01} = \phi_1 \cdot \theta t,\ f_{10} = \psi_1 \cdot i\theta t.$$

C.36. For each point y in X, choose a path a_y from x to y, and map a path γ in $\pi(X)_1$ from y to z to the homotopy class of $a_y \cdot \gamma \cdot a_z^{-1}$.

C.37. Choose $x_0 \in X_0$, and let $G = \mathrm{Aut}(x_0)$. Then $BG_\bullet$ is a subgroupoid of $X_\bullet$. Map $X_\bullet$ to $BG_\bullet$ by choosing $a_x \in X_1$ with $s(a_x) = x_0$, $t(a_x) = x$, with $a_{x_0} = e(x_0)$, and sending $b \in X_1$ to $a_x \cdot b \cdot {a_y}^{-1}$. The map $x \mapsto a_x$ is a 2-isomorphism from the composite $X_\bullet \to BG_\bullet \to X_\bullet$ to the identity on $X_\bullet$.

C.41. If α is a 2-isomorphism from $\phi'_\bullet \phi_\bullet$ to $1_{X_\bullet}$ and β is a 2-isomorphism from $\psi'_\bullet \psi_\bullet$ to $1_{Y_\bullet}$, then $\theta(x) = \phi'_1 \beta \phi_0(x) \cdot \alpha(x)$ defines a 2-isomorphism θ from $\phi'_\bullet \psi'_\bullet \psi_\bullet \phi_\bullet$ to $1_{X_\bullet}$. In the language of 2-categories, this is the composite of $1_{\phi'_\bullet} * \beta * 1_{\phi_\bullet}$ from $\phi'_\bullet \psi'_\bullet \psi_\bullet \phi_\bullet$ to $\phi'_\bullet 1_{Y_\bullet} \phi_\bullet = \phi'_\bullet \phi_\bullet$ and α from $\phi'_\bullet \phi_\bullet$ to $1_{X_\bullet}$.

C.42. Explicit isomorphisms between $G \ltimes (X/H)$ and $(G\backslash X) \rtimes H$, and 2-isomorphisms between their composites and the identities, can be constructed from choices of section of the maps $X \to X/H$ and $X \to G\backslash X$. See Exercise C.47.

C.48. When Z has at most two elements.

C.49. (a) Each is equivalent to the exactness of the sequence $0 \to V \to W \oplus V' \to W' \to 0$, the first taking v to $(L(v), \phi_V(v))$, the second taking (w, v') to $\phi_W(w) - L(v')$. (b) A splitting of $\mathrm{Ker}(L) \to V$ determines an isomorphism of $\mathrm{Ker}(L) \ltimes \mathrm{Coker}(L)$ to $V \ltimes W$, to which (a) applies; and similarly for a splitting $W \to \mathrm{Coker}(L)$. Without any splitting (for example for abelian groups), they are isomorphic because they both have components indexed by $\mathrm{Coker}(L)$, and all isotropy groups are $\mathrm{Ker}(L)$.

C.50. Apply the proposition.

C.53. Here $s = p_1$ and $t = p_2$ are the two projections from X_1 to X_0, with θ given by the identity map on X_1. And $X_2 = X_1 \, {}_t\!\times_{X_0,s} X_1$, with $q_1(a,b) = s(a)$, $q_2(a,b) = t(a) = s(b))$, $q_3(a,b) = t(b)$, $p_{12}(a,b) = a$, $p_{23}(a,b) = b$, $p_{13}(a,b) = a \cdot b$. Each θ_{ij} is given by a map from X_2 to X_1; in fact $\theta_{ij} = p_{ij}$. Each α_{ij}, α_{ji}, and α_i is an identity. A morphism $u_\bullet \colon X_0 \to Z_\bullet$ is given by map $u_0 \colon X_0 \to Z_0$, and $\tau : u_0 \circ s \overset{\tau}{\Rightarrow} u_0 \circ t$ is given by a map $\tau \colon X_1 \to Z_1$ with $s\tau = u_0 s$, $t\tau = u_0 t$, and $\tau(a \cdot b) = \tau(a) \cdot \tau(b)$. The required $v_\bullet \colon X_\bullet \to Z_\bullet$ is defined by $v_0 = u_0$ and $v_1 = \tau$; and $\rho \colon u_\bullet \Rightarrow v_\bullet \circ \pi$ is given by the map $e \circ u_0 \colon X_0 \to Z_1$. For the uniqueness, if $v'_\bullet \colon X_\bullet \to Z_\bullet$ and $\rho' \colon u_\bullet \Rightarrow v'_\bullet \circ \pi$ are others, the 2-isomorphism $\zeta \colon v_\bullet \Rightarrow v'_\bullet$ is given by the map $\zeta = \rho' \colon X_0 \to Z_1$.

C.55. The 2-isomorphism is given by the mapping

$$\theta \colon X \times G \longrightarrow X \times G \times G, \qquad (x, g) \mapsto (xg, g^{-1}, g).$$

C.59. Given $\phi_\bullet\colon X_\bullet \to Y_\bullet$, take $Y'_0 = X_0 \times Y_0$, and define $Y'_\bullet$ to be the pullback of $Y_\bullet$ by means of the projection $X_0 \times Y_0 \to Y_0$, so $Y'_1 = Y_1 \times X_0 \times X_0$. Map X_0 to Y'_0 by the graph of ϕ_0 and X_1 to Y'_1 by $a \mapsto (\phi_1(a), s(a), t(a))$.

C.61. The equivalence of (ii) to (v) follows from Exercise C.40; that (i) implies (ii) follows from the construction of the fibered product $X_\bullet \times_{Y_\bullet} T$; that (ii) implies (i) is proved by taking $T = Y_0$ and ψ_0 the identity.

C.62. Factor the morphism into $X_\to\colon Y'_\bullet \to Y_\bullet$ as in Exercise C.59. Let $Z_0 = Y'_0 = X_0 \times Y_0$, and let Z_1 be the image of $X_1 \to Y'_1$. The canonical map from $X_\bullet$ to $Z_\bullet$ is gerbe-like, and the canonical map $Z_\bullet \to Y'_\bullet$ (and hence $Z_\bullet \to Y'_\bullet \to Y_\bullet$) is representable.

C.63. The equivalence of (i) and (ii) follows from the construction of fibered products, and the equivalence of (ii) and (iii) from Exercise C.37.

C.64. If $Y_\bullet$ is the restriction, with $Y_0 = X_n$, then Y_1 consists of triples (a, b, c) with $a, b \in X_n$, $c \in X_1$, $s(c) = s(a_1)$, and $t(c) = s(b_1)$. Let $Z_\bullet = X_\bullet\{n\}$. Map $Y_\bullet$ to $Z_\bullet$ by the identity $Y_0 = X_n = Z_0$ and map $Y_1 \to Z_1$ by $(a, b, c) \mapsto (\phi_0, \ldots, \phi_n)$, where $\phi_0 = c$ and $\phi_i = a_i^{-1}\cdot \ldots \cdot a_1^{-1}\cdot c\cdot b_1\cdot \ldots \cdot b_i$ for $1 \le i \le n$.

C.65. The product in $V_\bullet^{(n)}$ is defined by

$$(a_1, \ldots, a_n, c, b_1, \ldots, b_n)\cdot(b_n^{-1}, \ldots, b_1^{-1}, d, e_1, \ldots, e_n) = (a_1, \ldots, a_n, c\cdot d, e_1, \ldots, e_n).$$

Set $Z_\bullet = X_\bullet\{n\}$. Map $V_\bullet = V_\bullet^{(n)}$ to $Z_\bullet$ by $V_0 = X_n = Z_0$ and V_1 to Z_1 by $(a_1, \ldots, a_n, c, b_1, \ldots, b_n) \mapsto (\phi_0, \ldots, \phi_n)$, where $\phi_0 = c$ and $\phi_i = a_{n+1-i}\cdot \ldots \cdot a_n\cdot c\cdot b_1\cdot \ldots \cdot b_i$ for $1 \le i \le n$. There is a canonical isomorphism between $(V_\bullet^{(n)})^{(m)}$ and $V_\bullet^{((n+1)(m+1)-1)}$, both having objects identified with X_{mn+m+n} and arrows identified with $X_{2(mn+m+n)+1}$.

C.66. (1) The identity e takes $(a_1, \ldots, a_n)$ to $(1_{sa_1}, a_1, \ldots, a_n)$ and the inverse i takes $(a_1, \ldots, a_{n+1})$ to $(a_1^{-1}, a_1\cdot a_2, a_3, \ldots, a_{n+1})$. (2) Let $Z_\bullet$ be the cross product groupoid, so $Z_0 = X_n = W_0^{(n)}$, and $Z_1 = \{((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \mid a_i = b_i \text{ for } i > 1\}$. Map Z_1 to $W_1^{(n)}$ by sending $((a_1, \ldots, a_n), (b_1, \ldots, b_n))$ to $(a_1\cdot b_1^{-1}, b_1, \ldots, b_n)$. (3) We have $Z_0 = X_n = (X_\bullet[n])_0$, and $Z_1 \to (X_\bullet[n])_1$ by

$$((a_1, \ldots, a_n), (b_1, \ldots, a_b)) \mapsto (\phi_0, \ldots, \phi_n, a_1, \ldots, a_n, b_1, \ldots, b_n),$$

with $\phi_0 = a_1\cdot b_1^{-1}$ and $\phi_i = 1_{sa_i}$ for $1 \le i \le n$.

C.67. Consider the morphism $W_\bullet^{(n+1)} \to W_\bullet^{(n)}$ that omits the last object on objects and arrows. This is easily checked to be square.

C.68. A strict isomorphism $\phi_\bullet$ from $X_\bullet(n, k)$ to $X_\bullet(n, k+1)$ is given by

$$\phi_1(a_1, \ldots, a_n) = (a_n^{-1}\cdot \ldots \cdot a_1^{-1}, a_1, \ldots, a_{n-1}),$$

with $\phi_0(a_1, \ldots, a_{n-1}) = (a_{n-1}^{-1}\cdot \ldots \cdot a_1^{-1}, a_1, \ldots, a_{n-2})$. [Should we omit this exercise?]

C.71. For $k = 0$, and $\sigma_1 = (b_1, \dots, b_{n-1})$ and $\sigma_2 = (c_1, \dots, c_{n-1})$, the other σ_i are determined, and one may take $\sigma = (c_1, c_1^{-1} \cdot b_1, b_2, \dots, b_{n-1})$. For $k = 1$, $\sigma_0 = (a_1, \dots, a_{n-1})$ and $\sigma_2 = (c_1, \dots, c_{n-1})$, take $\sigma = (c_1, a_1, a_2, \dots, a_{n-1})$. For $k > 1$, $\sigma_0 = (a_1, \dots, a_{n-1})$ and $\sigma_1 = (b_1, \dots, b_{n-1})$, take $\sigma = (b_1 \cdot a_1^{-1}, a_1, a_2, \dots, a_{n-1})$.