CHAPTER 5

Deligne–Mumford stacks

In this chapter, we will continue to work with the category $\mathcal{S}$ of schemes over a fixed base scheme Λ, endowed with the étale topology.

Deligne and Mumford identified a class of stacks as algebraic stacks. These are known now as *Deligne–Mumford stacks.* They are all isomorphic to stacks of the form $[R \rightrightarrows U]$, where $R \rightrightarrows U$ is an groupoid scheme with étale structure morphisms and quasi-compact relative diagonal (which implies quasi-affine relative diagonal, so that $[R \rightrightarrows U]$ is a stack). Conversely, if $R \rightrightarrows U$ is any groupoid scheme with étale structure morphisms and quasi-compact relative diagonal, then $[R \rightrightarrows U]$ will be a Deligne–Mumford stack.

1. Representable morphisms

A morphism of CFGs $f\colon \mathfrak{X} \to \mathfrak{Y}$ is called *representable* if, after base change to any scheme, it becomes a morphism of schemes. If this morphism of schemes always possesses some property (like flat, or smooth, or separated), then we will say that f possesses the same property.

Definition 5.1. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of CFGs. If, for every scheme T and morphism $\underline{T} \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T}$ is isomorphic to a scheme, then we say that f is **representable**.

Example 5.2. Here are some examples of representable morphisms.

(1) If $f\colon X \to Y$ is a morphism of schemes, then $\underline{f}\colon \underline{X} \to \underline{Y}$ is a representable morphism, by Example 2.25(1).
(2) Let G be an algebraic group. We recall the morphism $\mathrm{triv}\colon \underline{\Lambda} \to BG$, that associates to each scheme the trivial G-torsor over the scheme. Then $\mathrm{triv}\colon \underline{\Lambda} \to BG$ is representable, by Example 2.25(3).
(3) The forgetful morphism from $\mathcal{M}_{g,1}$ to $\mathcal{M}_g$ is representable, by Example 2.25(4).

Definition 5.3. Let $\mathbf{P}$ be a property of morphisms of schemes $f\colon X \to Y$ that satisfies:

(1) $\mathbf{P}$ is *preserved by arbitrary base extension*, i.e., if f has property $\mathbf{P}$ and $Y' \to Y$ is an arbitrary morphism then $X \times_Y Y' \to Y'$ also has property $\mathbf{P}$.
(2) $\mathbf{P}$ is *local for the étale topology* on Y, i.e., if $\{Y_\alpha \to Y\}$ is an étale covering family, and each $X \times_Y Y_\alpha \to Y_\alpha$ has property $\mathbf{P}$, then f has property $\mathbf{P}$.

Then we say that a representable morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of CFGs has property $\mathbf{P}$ if, for any scheme T and any morphism $\underline{T} \to \mathfrak{Y}$, the morphism $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T} \to \underline{T}$, has property $\mathbf{P}$.

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Notice, that since $f\colon \mathfrak{X} \to \mathfrak{Y}$ is representable, $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T}$ is isomorphic to a scheme, and then $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T} \to \underline{T}$ can be identified with a morphism of schemes (see Example 2.9(1)). This morphism of schemes is required to have property $\mathbf{P}$ (for any scheme T and morphism $\underline{T} \to \mathfrak{X}$) in Definition 5.3.

EXAMPLE 5.4. If G is a smooth quasi-affine group scheme (over Λ), then $\mathrm{triv}\colon \underline{\Lambda} \to BG$ is smooth. If the group scheme G is étale and quasi-affine (e.g., a finite group), then $\mathrm{triv}\colon \underline{\Lambda} \to BG$ is étale. The forgetful morphism $\mathcal{M}_{g,1} \to \mathcal{M}_g$ (where $g \geq 2$) is smooth and proper.

The following proposition lists, for future reference, many of the properties that satisfy conditions (1) and (2) of Definition 5.3. The fact of being preserved under base change is, in each case, either part of the definition (e.g., universally open), an immediate consequence of the definition (e.g., surjective), or a standard fact from algebraic geometry (e.g., étale). We provide, in each case, the appropriate reference to EGA for the property being local in the étale topology. (Warning: usually the precise statement in EGA is that a property holds if it holds after a faithfully flat quasi-compact base change. Any étale morphism is flat and locally of finite presentation, hence is open, and so the EGA statements plus the fact that these properties are local for the Zariski topology imply that they are local for the étale topology.)

PROPOSITION 5.5. *The following properties* $\mathbf{P}$ *of morphisms of schemes satisfy conditions (1) and (2) of Definition 5.3.*

(i) *surjective, radiciel (universally injective), universally bijective* [**EGA** IV.2.6.1]*,*
(ii) *universally open, universally closed, quasi-compact* [**EGA** IV.2.6.4]*,*
(iii) *separated, quasi-separated, locally of finite type, locally of finite presentation, finite type, finite presentation, proper, flat, an open embedding, a closed embedding, an isomorphism, a monomorphism* [**EGA** IV.2.5.1, IV.2.7.1]*,*
(iv) *an open embedding with dense image, a locally closed embedding,*
(v) *affine, quasi-affine, finite, quasi-finite* [**EGA** IV.2.7.1]*,*
(vi) *locally of finite type with fibers of dimension* $\leq d$*, locally of finite type with fibers (empty or) of pure dimension* d [**EGA** IV.4.1.4]*,*
(vii) *geometrically connected/reduced/irreducible fibers* [**EGA** IV.4.5.6, IV.4.6.10]*,*
(viii) *locally of finite type with geometrically Cohen–Macaulay/normal/regular fibers* [**EGA** IV.6.7.8]*,*
(ix) *formally unramified* [**EGA** IV.16.4.5, IV.17.2.1]*,*
(x) *unramified, smooth, étale* [**EGA** IV.17.7.4].

Only part (iv) of Proposition 5.5 requires further justification, given below. Our treatment of locally closed embeddings uses a preliminary lemma. Notice that [**EGA** IV.2.7.1] includes an argument applicable to quasi-compact embeddings, making use of the scheme-theoretic image (the smallest closed subscheme of the target through which a given morphism can be factored [**EGA** I.9.5.3]); without quasi-compactness we cannot appeal to the existence result [**EGA** I.9.5.1, IV.1.7.8] for the scheme-theoretic image.

Given a locally closed embedding of schemes $i\colon X \to Y$, there is a well-defined largest Zariski open subset $U \subset Y$ that contains the image of i as a closed subset. Precisely, U is the complement in Y of the set $\overline{i(X)} \smallsetminus i(X)$. Regarding U as an open subscheme, the morphism i then factors as a closed embedding of schemes $X \to U$ followed by the open embedding $U \to Y$. (See [**EGA** I.4.1.3, I.4.2.1].)

LEMMA 5.6. *Let $i\colon X \to Y$ be a locally closed embedding of schemes. Let $g\colon Y' \to Y$ be a morphism of schemes that is flat and locally of finite presentation, and set $X' = X \times_Y Y'$, with map $i'\colon X' \to Y'$ (also a locally closed embedding). Let $U = Y \smallsetminus \big(\overline{i(X)} \smallsetminus i(X)\big)$ and $U' = Y' \smallsetminus \big(\overline{i'(X')} \smallsetminus i'(X')\big)$. Then $U' = g^{-1}(U)$.*

PROOF. We have $\overline{i'(X')} \subset g^{-1}\big(\overline{i(X)}\big)$, hence

$$U' \supset g^{-1}(U).$$

The image $g(U')$ is open in Y, because a morphism that is flat and locally of finite presentation is open. We observe that there is a fiber diagram

$$\begin{array}{ccc} X' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ X & \longrightarrow & g(U') \end{array}$$

in which the vertical maps are faithfully flat and locally of finite presentation. Since the top horizontal map is a closed embedding, it follows that the bottom horizontal map is also a closed embedding. Hence $g(U')$ is contained in U. So

$$U' \subset g^{-1}(g(U')) \subset g^{-1}(U),$$

and we have the desired result. ☐

PROOF OF PROPOSITION 5.5. We treat part (iv). We already know that for a morphism to be an open embedding is preserved by base change and local for the étale topology. The corresponding statements for an open embedding with dense image now follow immediately from the fact that étale morphisms are open. It remains only to treat locally closed embeddings. Let $f\colon X \to Y$ be a morphism of schemes, let $g\colon Y' \to Y$ be an étale cover, and denote by f' the morphism from $X' = X \times_Y Y'$ to Y' obtained by base change. We suppose that f' is a locally closed embedding.

We set $Y'' = Y' \times_Y Y'$, with projections p_1, $p_2\colon Y'' \to Y'$, and denote by f'' the morphism from $X'' = X \times_Y Y''$ to Y''. We define $U' = Y' \smallsetminus \big(\overline{f'(X')} \smallsetminus f'(X')\big)$ and $U'' = Y'' \smallsetminus \big(\overline{f''(X'')} \smallsetminus f''(X'')\big)$. Lemma 5.6 implies that

$$p_1^{-1}(U') = U'' = p_2^{-1}(U').$$

Hence, there exists a unique open subscheme $U \subset Y$ such that $g^{-1}(U) = U'$.

We have a commutative diagram

$$\begin{array}{ccccc} X' & \longrightarrow & U' & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & U & \longrightarrow & Y \end{array}$$

in which the right-hand square and the outer square are cartesian, so the left-hand square is also cartesian. Since $X' \to U'$ is a closed embedding and $U' \to U$ is faithfully flat and locally of finite presentation, it follows that $X \to U$ is a closed embedding. Hence f, which is the composite bottom morphism in the diagram, is a locally closed embedding. □

Exercise 5.1. Give two examples of properties of morphisms of schemes which do not satisfy conditions (1) and (2).

Given a stack $\mathfrak{X}$, we of course have the forgetful functor to the base category $\mathfrak{X} \to \mathcal{S}$. Identifying $\mathcal{S}$ with $\underline{\Lambda}$, we can regard this as a morphism

$$f\colon \mathfrak{X} \to \underline{\Lambda}. \tag{5}$$

This morphism f is representable if and only if $\mathfrak{X}$ is isomorphic to $\underline{X}$ for some scheme X. So we may speak of a stack $\mathfrak{X}$ as being representable, meaning that it is isomorphic to a scheme, or equivalently, that the morphism (5) is representable.

Example 5.7. Here are some examples of representable stacks.

(1) The CFG $\underline{\mathrm{Hilb}}_{g,r}$ of Example 2.14 ($g \geq 2$, $r \geq 1$) is a representable stack.
(2) The stack $\overline{\mathcal{M}}_{0,n}$ is representable (for $n \geq 3$).

To show that a given stack $\mathfrak{X}$ is isomorphic to some $\underline{X}$, there is an easy initial reduction step to the case of families over affine base schemes. For if families over an affine base scheme, up to unique isomorphism, are identified with morphisms from the base scheme to X, then a patching argument gives the same result for an arbitrary base scheme.

Let $\mathfrak{X}$ be the stack $\underline{\mathrm{Hilb}}_{g,r}$ or $\overline{\mathcal{M}}_{0,n}$, and let $g \geq 2$, $n = 0$ in case (1) and $g = 0$, $n \geq 3$ in case (2). Let $S = \mathrm{Spec}(A)$ be an affine scheme, with family of stable n-pointed genus g curves $\pi\colon C \to S$. We suppose, in case (1) that we are also given an invertible sheaf $\mathcal{L}$ on S and an N-tuple of generating sections of $\omega_{C/S}^{\otimes r} \otimes \pi^*\mathcal{L}$, where $N = (2r-1)(g-1)$.

By [**EGA** IV.8.9.1], there exists a finitely-generated subring A_0 of A with a finite-type morphism $\pi_0\colon C_0 \to S_0 = \mathrm{Spec}(A_0)$, and in case (1) also an invertible sheaf $\mathcal{L}_0$ on S_0, such that there is an isomorphism $C \cong C_0 \times_{S_0} S$ (over S), and in case (1) also an isomorphism of $\mathcal{L}$ with the pullback of $\mathcal{L}_0$. Consider now the inductive system of subrings $A_\lambda \subset A$, finitely generated and containing A_0, and set $S_\lambda = \mathrm{Spec}(A_\lambda)$ and $C_\lambda = C_0 \times_{S_0} S_\lambda$. Let $C'_0 \to S_0$ be another morphism of finite type, and set $C'_\lambda = C'_0 \times_{S_0} S_\lambda$ and $C' = C'_0 \times_{S_0} S$. Then, by [**EGA** IV.8.8.2(i)] we have a bijection of sets

$$\varinjlim \mathrm{Hom}_{S_\lambda}(C'_\lambda, C_\lambda) \xrightarrow{\sim} \mathrm{Hom}_S(C', C).$$

Other results about projective limits of schemes ([**EGA** IV.8.5.4, IV.8.10.5, IV.11.2.6]) tell us that sections of a line bundle, and properties such as properness and flatness, come from $C_\lambda \to S_\lambda$ for suitable λ. Replacing A_0 with such A_λ, we may therefore assert that $C_0 \to S_0$ is a family of stable n-pointed genus g curves; in case (1) we have generating sections $\sigma_1, \ldots, \sigma_N \in \Gamma(C_0, \omega_{C_0/S_0}^{\otimes r} \otimes \pi_0^*\mathcal{L}_0)$ inducing a closed embedding $C_0 \to \mathbb{P}_{S_0}^{N-1}$; and the pullback to S can be identified with the given object of $\mathfrak{X}$ over S.

So we are reduced to the case of an affine Noetherian base scheme, and now we can apply Hilbert scheme machinery to assert that $\underline{\mathrm{Hilb}}_{g,r}$ is representable (by a suitable subscheme of the Hilbert scheme of $\mathbb{P}^{N-1}$).

More machinery (moduli spaces, etc.) is required for the assertion that $\overline{\mathcal{M}}_{0,n}$ is representable, so we restrict ourselves to a single case, $n=4$, and show that $\overline{\mathcal{M}}_{0,4} \cong \underline{\mathbb{P}^1}$. We still use the relative dualizing sheaf, for which an elementary description can be found in [**64**, Defn. 6.4.7, Exer. 6.4.5, 6.4.6]. Now $\omega_{C/S}(s_1+s_2+s_3+s_4)$ is relatively ample by by [**EGA** III.4.7.1] (since it is ample on fibers). Considering the twist by decreasing number of sections (4, down to 1) we see that on fibers the dimension of the space of sections goes down by 1 each time a section is omitted. The geometric consequence is that the resulting embedding into a 2-dimensional projective space bundle has the property that the 4 sections are mapped to 4 points in general position in $\mathbb{P}^2$. Here, "in general position" means that no 3 of the points lie on a line. It is a standard fact that any ordered collection of 4 points in general position in $\mathbb{P}^2$ are sent by a unique projective linear transformation to some chosen collection of 4 points:

$$[1,0,0], \quad [0,1,0], \quad [0,0,1], \quad [1,1,1].$$

Applying this projective linear transformation, the resulting family of conics (each smooth or a union of 2 lines) must be defined by an equation of the form

$$Axy + Bxz + Cyz = 0$$

with $A+B+C=0$. This leads to a canonical isomorphism

$$\overline{\mathcal{M}}_{0,4} \cong \underline{\mathrm{Proj}(\mathbb{Z}[A,B,C]/(A+B+C))}.$$

An additional example of representable stack is the stack of smooth families of genus g curves ($g \geq 2$) with Jacobi level n structure, for $n \geq 3$ (Example 2.15). Then the representability is a result of Serre (cf. Appendix to Grothendieck[1]).

The next result gives some of the formal properties of representable morphisms of stacks.

PROPOSITION 5.8. *Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{X}'$, $\mathfrak{Y}'$, and $\mathfrak{Z}$ be stacks.*

(i) *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ are representable morphisms, then $g \circ f\colon \mathfrak{X} \to \mathfrak{Z}$ is representable.*
(ii) *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is representable and $\mathfrak{Y}$ is representable, then $\mathfrak{X}$ is representable.*
(iii) *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is representable and $\mathfrak{Y}' \to \mathfrak{Y}$ is an arbitrary morphism, then $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ is representable.*
(iv) *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $f'\colon \mathfrak{X}' \to \mathfrak{Y}'$ are representable, then $f \times f'\colon \mathfrak{X} \times \mathfrak{X}' \to \mathfrak{Y} \times \mathfrak{Y}'$ is representable.*
(v) *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ are morphisms such that g and $g \circ f$ are representable, then f is representable.*

[1] J.-P. Serre, Rigidité du foncteur de Jacobi d'échelon $n \geq 3$, appendix to A. Grothendieck, Techniques de construction en géometrie analytique, X: Construction de l'espace de Teichmüller, Seminaire H. Cartan, 13e année (1960/61) Fasc 2, no. 17, Secrétariat mathématique, Paris, 1962.

PROOF. For (i), we have $\mathfrak{X} \times_{\mathfrak{Z}} \underline{T} \cong \mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \underline{T})$ (Exercise 2.6). The case $\mathfrak{Z} = \underline{\Lambda}$ of (i) is (ii). Another application of Exercise 2.6 gives (iii). For (iv), we use the isomorphism $(\mathfrak{X} \times \mathfrak{X}') \times_{\mathfrak{Y} \times \mathfrak{Y}'} \underline{T} \cong (\mathfrak{X} \times_{\mathfrak{Y}} \underline{T}) \times_{\underline{T}} (\mathfrak{X}' \times_{\mathfrak{Y}'} \underline{T})$ (Exercise $(\mathfrak{X} \times \mathfrak{X}') \times_{\mathfrak{Y} \times \mathfrak{Y}'} \mathfrak{Z} \cong (\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}) \times_{\mathfrak{Z}} (\mathfrak{X}' \times_{\mathfrak{Y}'} \mathfrak{Z})$ belongs in Section 2.5). For (v), given $\underline{T} \to \mathfrak{Y}$ we have the following 2-cartesian diagram

$$\begin{array}{ccccc}
\mathfrak{X} \times_{\mathfrak{Y}} \underline{T} & \longrightarrow & \underline{T} & & \\
\downarrow & & \downarrow & & \\
\mathfrak{X} \times_{\mathfrak{Z}} \underline{T} & \longrightarrow & \mathfrak{Y} \times_{\mathfrak{Z}} \underline{T} & \longrightarrow & \underline{T} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{X} & \longrightarrow & \mathfrak{Y} & \longrightarrow & \mathfrak{Z}
\end{array} \tag{6}$$

where the right-hand vertical map is the composite $\underline{T} \to \mathfrak{Y} \to \mathfrak{Z}$. The hypothesis implies that the stacks in the second row are all representable. So $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T}$ is isomorphic to a fiber product of schemes, hence is also representable. □

EXERCISE 5.2. If a representable morphism $\mathfrak{X} \to \mathfrak{Y}$ has property **P** (for some property **P** satisfying the conditions of Definition 5.3) and $\mathfrak{Y}' \to \mathfrak{Y}$ is an arbitrary morphism, then show that $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ also has property **P**.

PROPOSITION 5.9. *Let* **P** *be a property of morphisms of schemes that is preserved by arbitrary base extension and local for the étale topology. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a representable morphism of stacks. Let $g\colon \mathfrak{Y}' \to \mathfrak{Y}$ be a morphism of stacks, and assume that for every object y of $\mathfrak{Y}$, over a scheme T, there exists an étale cover $T' \to T$ such that the pull-back of y to T' is isomorphic to an object in the image of g. Then f has property* **P** *if and only if the morphism $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ obtained by base change has property* **P**.

PROOF. The forward implication is taken care of by Exercise 5.2. For the reverse implication, we let $f'\colon \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ denote the morphism obtained by base change, and we suppose that f' has property **P**. If T is a scheme and $\underline{T} \to \mathfrak{Y}$ is a morphism, we have to verify that $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T} \to \underline{T}$ has property **P**.

By hypothesis there is an étale cover $T' \to T$ such that the composite morphism $\underline{T'} \to \mathfrak{Y}$ factors, up to 2-isomorphism, as a morphism $\underline{T'} \to \mathfrak{Y}'$ followed by g. Then we have a diagram where the squares are 2-cartesian

$$\begin{array}{ccc}
\mathfrak{X} \times_{\mathfrak{Y}} \underline{T'} & \longrightarrow & \underline{T'} \\
\downarrow & & \downarrow \\
\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' & \xrightarrow{f'} & \mathfrak{Y}' \\
\downarrow & & \downarrow{\scriptstyle g} \\
\mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Y}
\end{array}$$

The morphism f' has property **P**. Hence so does the top map. Since **P** is local for the étale topology it follows that $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T} \to \underline{T}$ has property **P**. $\square$

Proposition 5.9, applied in the case that $\mathfrak{Y}' = \underline{U}$, a representable stack, tells us that to test whether the morphism f has property **P** it may be enough to check whether the single morphism f' (now a morphism of schemes) has property **P**.

PROPOSITION 5.10. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a representable morphism of stacks. Let $g\colon \mathfrak{Y}' \to \mathfrak{Y}$ be a representable surjective morphism of stacks. Then f is surjective if and only if the morphism $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ obtained by base change is surjective.*

PROOF. The proof is similar to the proof of Proposition 5.9. Suppose $f'\colon \mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{Y}' \to \mathfrak{Y}'$ is surjective. For arbitrary $\underline{T} \to \mathfrak{Y}$ we have to show that $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T} \to \underline{T}$ is surjective. Let T' be a scheme, with $\underline{T'} \cong \underline{T} \times_{\mathfrak{Y}} \mathfrak{Y}'$. So $T' \to T$ is surjective. The hypothesis implies that $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T'} \to \underline{T'}$ is surjective. So the composite $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T'} \to \underline{T}$ is surjective as well. Since this factors through $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T}$ it follows that $\mathfrak{X} \times_{\mathfrak{Y}} \underline{T} \to \underline{T}$ is surjective. $\square$

REMARK 5.11. The definition of representable morphism (and representable stack) given in this section will remain valid throughout Part I of the book. However, in Part II we will need to re-define the notion of representable morphism, replacing schemes by algebraic spaces. Algebraic spaces will not appear, however, until later in Part I. Once we have algebraic spaces at our disposal, we will then have two notions of representability: *representable morphisms* will be representable by algebraic spaces, and then morphisms which satisfy the condition given in Definition 5.1 (representability by schemes) will be called *strongly representable*. An important point will be that the definition of Deligne–Mumford stack, which uses the notion of representable morphism, *does not change* when we switch to the new notion of representability in Part II.

2. Stacks with representable diagonal

Given a stack $\mathfrak{X}$, we are particularly interested in properties of the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$. There are two motivations for this. First, there was the important role of the (relative) diagonal in the setting of groupoid schemes, that we saw in the previous chapter. Second, the diagonal produces, via base change, arbitrary fiber products over $\mathfrak{X}$ (Exercise 2.7). The central role played by the fiber product in algebraic geometry is reflected in the approach (taken by Deligne and Mumford) to the definition of algebraic stack, which assigns particular importance to the diagonal of a stack.

PROPOSITION 5.12. *Given a stack $\mathfrak{X}$, the following are equivalent.*

(1) *The diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable.*
(2) *For arbitrary schemes T and U and morphisms $\underline{T} \to \mathfrak{X}$ and $\underline{U} \to \mathfrak{X}$ the fiber product $\underline{T} \times_{\mathfrak{X}} \underline{U}$ is representable.*
(3) *Any morphism $\underline{T} \to \mathfrak{X}$ (where T is a scheme) is representable.*
(4) *For any scheme T and pair of morphisms $x\colon \underline{T} \to \mathfrak{X}$ and $y\colon \underline{T} \to \mathfrak{X}$ the fiber product $\underline{T}\ {}_x{\times}_{\mathfrak{X},y}\, \underline{T}$ is representable.*
(5) *For any scheme T and pair of objects x, y of $\mathfrak{X}_T$, the sheaf $\mathcal{I}som_{\mathfrak{X}}(x,y)$ is representable by a scheme over T.*

PROOF. We have (1) implies (2), by Exercise 2.7, which gives $\underline{U}\times_{\mathfrak{X}}\underline{V} \cong \underline{U \times V}\times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}$. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate. For (4) $\Rightarrow$ (5), we have the following description of the stack associated with the sheaf $\mathcal{I}som_{\mathfrak{X}}(x, y)$:

$$\underline{\mathcal{I}som}_{\mathfrak{X}}(x, y) \cong (\underline{T}\ {}_x\times_{\mathfrak{X},y} \underline{T}) \times_{\underline{T\times T}} \underline{T}.$$

Lastly, by the 2-cartesian diagram

$$\begin{array}{ccc} \underline{\mathcal{I}som}_{\mathfrak{X}}(x, y) & \longrightarrow & \underline{T} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X}\times\mathfrak{X} \end{array}$$

we have (5) implies (1). □

REMARK 5.13. Let $\mathfrak{X}$ be a stack with representable diagonal, and let $t\colon \underline{T} \to \mathfrak{X}$ $u\colon \underline{U} \to \mathfrak{X}$ be arbitrary morphisms. Then, working through the implications (5) $\Rightarrow$ (1) $\Rightarrow$ (2) explicitly, we have

$$\underline{T} \times_{\mathfrak{X}} \underline{U} \cong \underline{\mathcal{I}som}_{\mathfrak{X}}(p_1^*t, p_2^*u)$$

where p_1 and p_2 denote projections from $T\times U$. In particular, the fiber product $\underline{T}\ {}_x\times_{\mathfrak{X},y}\underline{T}$ appearing in (4) is $\underline{\mathcal{I}som}_{\mathfrak{X}}(p_1^*x, p_2^*y)$.

We have all the ingredients in place in order to state the definition of Deligne–Mumford stack.

DEFINITION 5.14. A stack $\mathfrak{X}$ is a **Deligne–Mumford stack** if it satisfies the following two properties:

(1) The diagonal $\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ is representable, and is quasi-compact and separated.
(2) There exists a scheme U and a morphism $\underline{U} \to \mathfrak{X}$ which is étale and surjective.

Notice that once the diagonal is asserted to be representable, it makes sense to describe it as quasi-compact and separated (both of these are valid properties **P** according to Proposition 5.5). Also, as a consequence of having representable diagonal, the morphism $\underline{U} \to \mathfrak{X}$ is representable (by Proposition 5.12), hence it is sensible to speak of this morphism as being étale and surjective.

Here are some results that can be used in some cases to simplify the verification of (1) and (2).

PROPOSITION 5.15. *Let $\mathfrak{X}$ be a stack. Suppose, for every scheme T and objects x and y of $\mathfrak{X}_T$ there exists an étale cover $f\colon T' \to T$ such that $\mathcal{I}som_{\mathfrak{X}}(f^*x, f^*y)$ is represented by a scheme, quasi-affine over T'. Then the diagonal $\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ is representable and quasi-affine. (So, in particular the diagonal is quasi-compact and separated.)*

PROOF. By hypothesis, there is a scheme Y' with quasi-affine morphism $Y' \to T'$, and an isomorphism $\mathcal{I}som_{\mathfrak{X}}(f^*x, f^*y) \cong h_{Y'}$ of sheaves on T', where $h_{Y'}$ denotes the functor of points of Y'. Set $T'' = T' \times_T T'$, with projections p_1 and p_2 to T' and morphism $F\colon T'' \to T$. Now we have isomorphisms $\mathcal{I}som_{\mathfrak{X}}(F^*x, F^*y) \cong h_{Y'\times_T T'}$ and $\mathcal{I}som_{\mathfrak{X}}(F^*x, F^*y) \cong h_{T'\times_T Y'}$ These give rise to an isomorphism $\varphi\colon Y'\times_T T' \to T'\times_T Y'$. Because of similar isomorphisms of sheaves over $T'\times_T T'\times_T T'$, we see that φ satisfies the cocycle condition. By Proposition A.17, there is a scheme Y with quasi-affine morphism $Y \to T$, and an isomorphism $\lambda\colon T' \times_T Y \to Y'$ such that $T' \times_T \lambda = \varphi \circ (\lambda \times_T T')$.

Since $\mathcal{I}som_{\mathfrak{X}}(x,y)$ and h_Y are both sheaves for the étale topology on T, we get an isomorphism $\mathcal{I}som_{\mathfrak{X}}(x,y) \cong h_Y$. Concretely, given $g\colon S \to T$ and $g^*x \cong g^*y$, we get, by pullback, an isomorphism over $S \times_T T'$, and this corresponds to a morphism $S \times_T T' \to Y'$. Consider the composite morphism $S \times_T T' \to Y' \cong T' \times_T Y \to Y$ with λ^{-1} and the second projection. Using the compatibility of λ it follows that the two morphisms $S \times_T T'' \to Y$ obtained by pullback are equal, hence there is an induced morphism $S \to Y$. This is a morphism of sheaves on T that, after restriction to T', becomes an isomorphism, hence is itself an isomorphism. □

Now we are prepared to give the first examples of Deligne–Mumford stacks.

Example 5.16. Let X be a scheme, and assume that X is quasi-separated (over the base scheme). Then $\underline{X}$ is a DM stack.

We recall that to be quasi-separated means to have quasi-compact diagonal (every separated scheme is quasi-separated, and so is every locally Noetherian scheme). That means that axiom (1) is satisfied. The identity map $\underline{X} \to \underline{X}$ is étale and surjective, so axiom (2) is satisfied.

Example 5.17. Let G be a finite group, or more generally a group scheme, étale and quasi-finite over the base scheme. Then BG is a DM stack. If G acts on a quasi-separated scheme X, then $[X/G]$ is a DM stack.

To show that Axiom (1) is satisfied, we use Proposition 5.15. Given a pair of G-torsors over a scheme T, we know they can be trivialized on some étale cover, so we are reduced to showing that $\mathcal{I}som_{\mathfrak{X}}(x,y)$ is representable in case both x and y are the trivial G-torsor. Then $\mathcal{I}som_{\mathfrak{X}}(x,y)$ is represented by the scheme $T \times G$, and this is quasi-affine over T. We claim that the morphism $\underline{\Lambda} \to BG$ corresponding to the trivial G-torsor over the base scheme satisfies the condition of Axiom (2). By Example 2.25(3), this morphism becomes, after base change, the morphism $E \to T$, where E is a G-torsor over T, hence is an étale cover of T.

Exactly the same argument takes care of Axiom (2) for $[X/G]$, using the morphism $\underline{X} \to [X/G]$. For Axiom (1) we are, as above, reduced to considering trivial G-torsors. Now a trivial G-torsor $T \times G$ with equivariant morphism to X is determined uniquely by a morphism $T \to X$ (this will be the restriction of the equivariant morphism to $T \times \{e_G\}$ and will determine the morphism $T \times G \to X$ by equivariance).

Exercise 5.3. Let f and g be a pair of morphisms $T \to X$, and let x and y denote the corresponding equivariant morphisms $T \times G \to X$. Then $\mathcal{I}som_{[X/G]}(x,y)$ is represented by a scheme, quasi-affine over T, and Axiom (1) for $[X/G]$ is satisfied.

Consider a stack $[X/G]$ where G is an algebraic group of positive dimension. In general, this will not be a DM stack, e.g., in the case $X = \Lambda$, so $[X/G] = BG$.

Exercise 5.4. Let G be an algebraic group (or group scheme) of positive dimension (over the base scheme). Then BG is not a DM stack.

However we will see that $[X/G]$ is a DM stack when G (assumed to be a group scheme, smooth and separated over Λ) acts on X with *finite, reduced geometric stabilizers*. We need to avoid the positive-dimensional stabilizers, e.g., of the trivial action of G on Λ in Exercise 5.4. In characteristic $p > 0$ it is also possible for stabilizers to be finite but non-reduced. (The action of the multiplicative group scheme $\mathbb{G}_m = \mathbb{A}^1 \smallsetminus \{0\}$ on itself by $t \cdot z = t^p z$ has stabilizer μ_p, the subgroup scheme of $\mathbb{G}_m$ defined by $t^p = 1$.) If we avoid this sort of phenomenon as well, then we get a DM stack. Showing this requires a criterion of Deligne and Mumford, which we will present later in this chapter.

In particular, the stack $\mathcal{M}_g$ is isomorphic to a stack of the form $[PGL_{5g-5} \backslash \operatorname{Hilb}_{g,3}]$ where $\operatorname{Hilb}_{g,3}$ is a locus in the Hilbert scheme of $\mathbb{P}^{5g-6}$ (see Example 1.2C). We will see that PGL_{5g-5} acts with finite reduced geometric stabilizers. The criterion of Deligne and Mumford will then show that $\mathcal{M}_g$ is a DM stack. (The same argument will apply to $\overline{\mathcal{M}}_g$.)

We will conclude this section with some formal statements about stacks with representable diagonal. These will be useful later, e.g., in showing that any fiber product of DM stacks is again a DM stack.

LEMMA 5.18. *Let $f \colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of stacks. If $\mathfrak{Y}$ has representable diagonal then the natural map $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable.*

PROOF. The morphism $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is obtained by base change from the representable diagonal morphism $\mathfrak{Y} \to \mathfrak{Y} \times \mathfrak{Y}$. □

PROPOSITION 5.19. (a) *If $\mathfrak{X} \to \mathfrak{Y}$ is a morphism of stacks with representable diagonal then the relative diagonal $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is representable.*

(b) *If $\mathfrak{X} \to \mathfrak{Y}$ is a representable morphism of stacks, and if $\mathfrak{Y}$ has representable diagonal, then $\mathfrak{X}$ has representable diagonal.*

(c) *If $f \colon \mathfrak{X} \to \mathfrak{Y}$ and $g \colon \mathfrak{Y} \to \mathfrak{Z}$ are two morphisms of stacks such that $g \circ f$ is representable and $\mathfrak{Y}$ and $\mathfrak{Z}$ have representable diagonal then f is representable.*

PROOF. The representable diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ factors as

$$\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}. \tag{7}$$

By Lemma 5.18 the morphism $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable. Thus by Proposition 5.8(v) $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is also representable. This proves (a).

If $\mathfrak{Y}$ has representable diagonal and $\mathfrak{X} \to \mathfrak{Y}$ is representable, then the identity morphism of $\mathfrak{X}$ factors as $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X}$, the relative diagonal followed by projection onto the second factor. The latter is obtained by base change from $\mathfrak{X} \to \mathfrak{Y}$, hence is representable. So the relative diagonal is also representable, by Proposition 5.8(v). Combining this observation with Lemma 5.18, we have the diagonal of $\mathfrak{X}$ expressed in (7) as a composite of representable morphisms. This establishes (b).

For (c), we factor f as

$$\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \to \mathfrak{Y}.$$

The first map is gotten by base change from $\mathfrak{Y} \to \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{Y}$, which is representable by (a). The second map comes via base change from $g \circ f$, and this is representable by hypothesis. So f is representable. □

3. Atlases for DM stacks

Here we make the dictionary between stacks and groupoids precise in the case of DM stacks. To every DM stack $\mathfrak{X}$ and every choice of étale atlas U (scheme with étale surjective morphism to $\mathfrak{X}$) there is an symmetry groupoid, and this will be an étale groupoid scheme with quasi-compact separated relative diagonal. Conversely, given such groupoid scheme the associated stack of torsors will be a DM stack.

Many of the proofs of facts about DM stacks start by considering a groupoid scheme presentation and then working with the groupoid scheme. Also, étale groupoid schemes are concrete objects (we have written down many étale groupoid schemes already in Chapters 1 through 4): if you write down an étale groupoid scheme then you get a concrete example of a DM stack.

PROPOSITION 5.20. *Let $\mathfrak{X}$ be a DM stack. Let U be a scheme, and u an object of $\mathfrak{X}_U$ such that the associated morphism $u\colon \underline{U} \to \mathfrak{X}$ is étale and surjective. Then, if we take R to be a scheme with $\underline{R} \cong \mathfrak{Sym}_{\mathfrak{X}}(u,u)$, the associated symmetry groupoid $R \rightrightarrows U$ has quasi-affine relative diagonal and there is an induced isomorphism $\mathfrak{X} \cong [R \to U]$.*

PROOF. Since $\mathfrak{X}$ has representable diagonal, the symmetry groupoid $\mathfrak{Sym}_{\mathfrak{X}}(u,u)$ is indeed isomorphic to $\underline{R}$ for a scheme R, and we have a groupoid scheme $s,\, t\colon R \rightrightarrows U$ by Proposition 3.5. The relative diagonal $R \to U \times U$ factors as

$$R \xrightarrow{\Delta_R} R \times R \xrightarrow{s\times t} U \times U \qquad (8)$$

where Δ_R denotes the diagonal of R. Both morphisms in (8) are locally quasi-finite [**EGA** $\mathrm{Err}_{\mathrm{III}}$.20], hence so is their composite. The composite $R \to U \times U$ is quasi-compact and separated, since the diagonal of $\mathfrak{X}$ has these properties. Now we apply [**EGA** IV.18.12.12], which says that a morphism that is quasi-finite (i.e., quasi-compact and locally quasi-finite) and separated is quasi-affine. So $R \to U \times U$ is quasi-affine.

We can now apply Proposition 4.19 to obtain an isomorphism $\mathfrak{X} \cong [R \rightrightarrows U]$. Indeed, since $\underline{U} \to \mathfrak{X}$ is étale and surjective, for an arbitrary scheme T with $x\colon \underline{T} \to \mathfrak{X}$ the morphism obtained by base change $\underline{U} \times_{\mathfrak{X}} \underline{T} \to \underline{T}$ is étale and surjective. Let T' be a scheme, with $\underline{T'} \cong \underline{U} \times_{\mathfrak{X}} \underline{T}$. By this isomorphism, there are maps $T' \to U$ and $T' \to T$, and an isomorphism in $\mathfrak{X}_{T'}$ between the pullbacks $x|_{T'}$ and $u|_{T'}$. With the étale cover $T' \to T$, the hypothesis of Proposition 4.19 is fulfilled. □

As a converse, we have that any étale groupoid scheme with quasi-compact separated relative diagonal gives rise to a DM stack. This arises as a corollary to the following statement.

PROPOSITION 5.21. *Let $s,\, t\colon R \rightrightarrows U$ be a groupoid scheme such that the relative diagonal $(s,t)\colon R \to U \times U$ is quasi-affine, and set $\mathfrak{X} = [R \rightrightarrows U]$. Then $\mathfrak{X}$ has representable diagonal. Moreover, if $\mathbf{P}$ is any property of morphisms of schemes that is stable under base change and local for the étale topology, then:*

(i) *The diagonal $\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ has property $\mathbf{P}$ if and only if $R \to U\times U$ has property $\mathbf{P}$.*
(ii) *The morphism $\underline{U} \to \mathfrak{X}$ (corresponding to the identity of U) has property $\mathbf{P}$ if and only if s (or t) has property $\mathbf{P}$.*

Proof. We may use Proposition 5.15 to check representability of the diagonal. By definition, objects of $\mathfrak{X}$ are étale locally trivial $(R \rightrightarrows U)$-torsors. So representability of the diagonal follows, if we can show that for an arbitrary scheme T and morphisms x, $y\colon T \to U$, $\mathcal{I}som_{\mathfrak{X}}(x,y)$ is representable by a scheme, quasi-affine over T. But this is isomorphic to $\mathcal{I}som_{[R\rightrightarrows U]^{\mathrm{pre}}}(x,y)$, since the stackification morphism is a fully faithful functor. Immediately from the definition, the latter is represented by the fiber product

$$T \times_{U\times U} R$$

of the morphism $(x,y)\colon T \to U \times U$ and the relative diagonal of the groupoid scheme. This is a scheme, quasi-affine over T. So $\mathfrak{X}$ has representable diagonal.

We may now apply Proposition 5.9 to the morphism $\underline{U} \times \underline{U} \to \mathfrak{X} \times \mathfrak{X}$, respectively to $\underline{U} \to \mathfrak{X}$, to obtain statement (i), respectively (ii). □

Corollary 5.22. *Let $R \rightrightarrows U$ be an étale groupoid scheme such that the relative diagonal $R \to U \times U$ is quasi-compact and separated. Set $\mathfrak{X} = [R \rightrightarrows U]$. Then $\mathfrak{X}$ is a DM stack, with étale surjective morphism $\underline{U} \to \mathfrak{X}$.*

Proof. The relative diagonal $R \to U \times U$ is, by hypothesis, quasi-compact and separated. By the argument of the proof of Proposition 5.20 involving the factorization (8), the relative diagonal is locally quasi-finite, hence quasi-affine.

Now by Proposition 5.21, $\mathfrak{X}$ is a stack with representable quasi-compact separated diagonal and étale surjective morphism $\underline{U} \to \mathfrak{X}$. □

By Proposition 4.20, the étale surjective morphism $u\colon \underline{U} \to \mathfrak{X}$ produced in Corollary 5.22 satisfies $\mathfrak{Sym}_{\mathfrak{X}}(u,u) \cong \underline{R}$, and the associated symmetry groupoid is the given groupoid scheme $R \rightrightarrows U$. In this way, the results in this section tell us how to go back and forth between DM stacks and étale groupoid schemes with quasi-compact separated relative diagonal.

Example 5.23. We have seen that $[X/G]$ is a DM stack if G is a group scheme, quasi-affine and étale over the base scheme, acting on a quasi-separated scheme X. The morphism $\underline{X} \to [X/G]$ is étale surjective (Example 5.17). A groupoid scheme presentation is the transformation groupoid $X \times G \rightrightarrows X$ (Example 3.9). We have $[X/G] \cong [X \times G \rightrightarrows X]$ (Proposition 4.19). This is the isomorphism $\mathfrak{X} \cong [R \rightrightarrows U]$ that results from taking $\mathfrak{X} = [X/G]$ and $U = X$ (with morphism corresponding to the trivial G-torsor) in Proposition 5.20.

The groupoid scheme associated to a DM stack depends on a choice of étale surjective map from a scheme. So there can be different groupoid schemes arising from the same DM stack. As an easy example of this, for any quasi-separated scheme X, we have seen that $\underline{X}$ is a DM stack, and for any étale surjective map $U \to X$ we have $\underline{X} \cong [R \rightrightarrows U]$ where $R = U \times_X U$. (In fact, in the previous chapter we observed that we have such an isomorphism.) The next exercise gives a more interesting example of atlases of a DM stack.

Exercise 5.5. Let X be a scheme and $Y \to X$ an unramified degree 2 cover. Let $G = S_3$, the symmetric group on 3 elements, and set $\mathfrak{X} = \underline{X} \times BG$.

(i) We have $\mathfrak{X} \cong [X \times G \rightrightarrows X]$, coming from the obvious étale morphism $\underline{X} \to \mathfrak{X}$.
(ii) There is another étale surjective morphism $\underline{X} \to \underline{X} \times BG$, given by the identity map on the first factor and the image of the $(\mathbb{Z}/2\mathbb{Z})$-torsor $Y \to X$ under the map $B(\mathbb{Z}/2\mathbb{Z}) \to BG$ coming from a nontrivial group homomorphism $\mathbb{Z}/2\mathbb{Z} \to G$ (Example 2.9(2)).
(iii) Taking R so that $\underline{R} \cong \underline{X} \times_{\mathfrak{X}} \underline{X}$, the fiber product of the morphism in (ii) with itself, we have $\mathfrak{X} \cong [R \rightrightarrows X]$, with R generally different from $X \times G$.

Remark 5.24. Let us examine the argument at the end of the proof of Proposition 5.20 more closely. We used the étale cover $u\colon \underline{U} \to \mathfrak{X}$ to obtain an étale cover of an arbitrary scheme T, given an object of $\mathfrak{X}$ over T, such that the object after pullback becomes isomorphic to a pullback of u. Simply, we formed the fiber product, and that gave us the étale cover of T.

We know, more generally, that a smooth surjective morphism of schemes admits sections étale locally. Precisely, if $Y \to X$ is a smooth surjective morphism of schemes, then there exists an étale cover $X' \to X$ and a morphism $X' \to Y$ over X [**EGA** IV.17.16.3(ii)]. Hence if $\mathfrak{X}$ is any stack and U a scheme, then any representable *smooth* surjective morphism $u\colon \underline{U} \to \mathfrak{X}$ has the property that any object of $\mathfrak{X}$ (over a scheme T) admits, after pullback to some étale cover (of the scheme T), a morphism to u in $\mathfrak{X}$.

Let us return to the isomorphism $\underline{X} \cong [R \rightrightarrows U]$, where U is any étale cover of a quasi-separated scheme X. By Remark 5.24, such an isomorphism still holds if $U \to X$ is smooth surjective. That is because the existence of étale local sections lets us apply Proposition 4.19, which tells us that the obvious morphism $[R \rightrightarrows U]^{\mathrm{pre}} \to \underline{X}$ give rise to such an isomorphism.

If the covering $U \to X$ is only flat, and not smooth, then $[R \rightrightarrows U]$ need not be isomorphic to $\underline{X}$. In fact, it need not be a DM stack at all, as we see in the next exercise. The subject of flat groupoid schemes is a delicate one, and to obtain good results, one has to work in the *fppf topology* (where covering families are morphisms, locally of finite presentation, whose images cover the target scheme). Flat groupoid schemes, and the fppf topology, are not needed in Part I of this book, but will make an appearance in Part II.

Exercise 5.6. Let $f\colon U \to X$ be a covering of smooth projective irreducible curves over the complex numbers, which is totally ramified above some point $x \in X$. (This means that $f^{-1}(x)$ consists of a single point $u \in U$.) Set $R = U \times_X U$ and $\mathfrak{X} = [R \rightrightarrows U]$. Then $\mathfrak{X}$ has representable separated quasi-compact diagonal, but there exists no étale surjective morphism $\underline{V} \to \mathfrak{X}$ for any scheme V.

4. A Criterion for a stack to be a Deligne-Mumford stack

Often, the most natural presentation of a Deligne–Mumford stack has a smooth, rather than an étale atlas. In this section we state and prove a criterion, due to Deligne and Mumford, that allows us to determine when stacks with smooth covers are in fact DM stacks.

First we need a preliminary result.

LEMMA 5.25. *Let $\mathfrak{X}$ be a stack such that the diagonal of $\mathfrak{X}$ is representable, separated, and quasi-compact. Assume there exists a surjective morphism $\underline{W} \to \mathfrak{X}$ with the property that for every closed point $w \in W$ there exists a scheme U_w and an étale morphism $\underline{U_w} \to \mathfrak{X}$ such that the projection morphism $\underline{U_w} \times_{\mathfrak{X}} \underline{W} \to \underline{W}$ has nonempty fiber over w. Then $\mathfrak{X}$ is a DM stack.*

PROOF. It remains only to exhibit a scheme U with étale surjective morphism $\underline{U} \to \mathfrak{X}$. We claim that we can take $U = \coprod U_w$ (disjoint union taken over all closed points $w \in W$). Since each $\underline{U_w} \to \mathfrak{X}$ is étale, it follows that $\underline{U} \to \mathfrak{X}$ is étale. To show it is surjective, we use Proposition 5.10, which tells us it is enough to verify that $\underline{U} \times_{\mathfrak{X}} \underline{W} \to \underline{W}$ is surjective. Since this is an étale morphism it is open, hence is suffices to show that every closed point of W is contained in the image. This is the case, since by hypothesis, for any closed point $w \in W$ the image of the morphism $\underline{U_w} \times_{\mathfrak{X}} \underline{W} \to \underline{W}$ contains w. $\square$

THEOREM 5.26. *Let $\mathfrak{X}$ be a stack, and assume that*

(1) *$\mathfrak{X}$ has representable, separated, and quasi-compact diagonal,*
(2) *there exists a scheme U and a morphism $\underline{U} \to \mathfrak{X}$ that is smooth and surjective,*
(3) *the diagonal of $\mathfrak{X}$ is formally unramified.*

Then $\mathfrak{X}$ is a DM stack.

Before we give the proof, we discuss the conditions appearing in the statement. We let R denote a scheme, with $\underline{R} \cong \underline{U} \times_{\mathfrak{X}} \underline{U}$, which must exists by condition (1). So we have a groupoid scheme $s, t \colon R \rightrightarrows U$. There is a basic 2-cartesian diagram

$$\begin{array}{ccc} \underline{R} & \xrightarrow{(s,t)} & \underline{U} \times \underline{U} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X} \times \mathfrak{X} \end{array} \tag{9}$$

The right-hand morphism in (9) is a product of representable morphisms, and hence is representable by Proposition 5.8(iv). It is smooth surjective since it is a product of smooth surjective morphisms. Now by Remark 5.24 we are able to apply Proposition 5.9, which tells us that under the assumption that $\mathfrak{X}$ satisfies (1) and (2) the remaining hypothesis (3) is equivalent to the hypothesis

(3′) *The morphism $R \to U \times U$ is formally unramified.*

Still under the assumption that $\mathfrak{X}$ satisfies (1) and (2), we observe that the factorization (8) of $R \to U \times U$ as the diagonal of R followed by the smooth morphism $s \times t$, gives us that the morphism $R \to U \times U$ is locally of finite type.

A morphism of schemes $f \colon X \to Y$ is formally unramified if and only if $\Omega_f = 0$ (cf. the Glossary). When f is locally of finite type, there are two additional equivalent characterizations. First is that for every closed point $y \in Y$ the fiber $f^{-1}(y)$ is discrete and reduced, and the residue field of every point of $f^{-1}(y)$ is a separable extensions of the residue field of y. Second is that f has discrete reduced fibers at all geometric points. Every (locally closed) embedding of schemes is unramified, as is every étale morphism. By definition, a morphism is unramified if it is formally unramified and locally of finite presentation; however, as pointed out in [**EGA** IV.17.4.10], many of the basic results concerning unramified morphisms apply as well to morphisms that are formally unramified and locally of finite type.

To make the link with stacks of torsors, we see that hypotheses (1), (2), and (3) imply that the relative diagonal $R \to U \times U$ is quasi-finite (since it is locally of finite type and quasi-compact, with discrete fibers) and separated, hence quasi-affine by [**EGA** IV.18.12.12]. Now Remark 5.24 lets us apply Proposition 4.19 to conclude that $\mathfrak{X} \cong [R \rightrightarrows U]$.

Remark 5.27. Conversely, by Proposition 5.21, if $R \rightrightarrows U$ is a smooth groupoid scheme with quasi-compact, separated, formally unramified relative diagonal, then $\mathfrak{X} = [R \rightrightarrows U]$ satisfies conditions (1), (2), and (3) of Theorem 5.26.

The criterion to be a DM stack in Theorem 5.26 is really an alternative characterization of DM stacks. That is because every DM stack satsifies (1), (2), and (3). Indeed, conditions (1) and (2) follows directly from the definition of DM stack. Condition (3′) holds for any DM stack $\mathfrak{X}$ with étale cover $\underline{U} \to \mathfrak{X}$ and $\underline{R} \cong \underline{U} \times_{\mathfrak{X}} \underline{U}$, because the factorization (8) expresses the relative diagonal as a composite of an embedding and an étale morphism.

Proof of Theorem 5.26. Vital for the proof are the basic cartesian diagrams arising from the groupoid scheme, which we recall here:

$$\begin{array}{ccc} R\,{}_t{\times}_s\,R & \xrightarrow{\mathrm{pr}_2} & R \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle s} \\ R & \xrightarrow[t]{} & U \end{array} \qquad \begin{array}{ccc} R\,{}_t{\times}_s\,R & \xrightarrow{m} & R \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle s} \\ R & \xrightarrow[s]{} & U \end{array} \qquad \begin{array}{ccc} R\,{}_t{\times}_s\,R & \xrightarrow{\mathrm{pr}_2} & R \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ R & \xrightarrow[t]{} & U \end{array}$$

We first observe that we can express U as a disjoint union $\coprod U_n$ of subschemes, such that $\underline{U}_n \to \mathfrak{X}$ is smooth of relative dimension n, for each $n \geq 0$; the argument makes use of these diagrams. We have $R = \coprod R_n$, where R_n is the locus (open and closed in R) of points of R where the morphism t has relative dimension n. Because formation of such a locus commutes with arbitrary base change, we have

$$\mathrm{pr}_1^{-1}(R_n) = \{\, p \in R\,{}_t{\times}_s\,R \mid \mathrm{pr}_2 \text{ has relative dimension } n \text{ at } p \,\} = m^{-1}(R_n)$$

Hence there exists a unique (open and closed) subscheme $U_n \subset U$ satisfying $s^{-1}(U_n) = R_n$. The U_n's cover U (because their pre-images by s cover R), and $\underline{U}_n \to \mathfrak{X}$ is smooth of relative dimension n (by Proposition 5.9 combined with Remark 5.24).

We denote by $x\colon \underline{U} \to \mathfrak{X}$ the given smooth cover. An outline for the proof, now, is as follows:

- Define a locally free sheaf Ω_x on U (which should be thought of as a sheaf of relative differentials of U over $\mathfrak{X}$) and a morphism $\varphi\colon \Omega_U \to \Omega_x$; under the hypotheses the morphism is surjective.

- For any closed point $w \in U$, show there is a Zariski open neighborhood of w on which the rank of Ω_x is constant and on which there exist global functions $f_1, \ldots, f_n$ (where n is the rank of Ω_x) such that Ω_x is spanned by the df_i; shrinking U to this neighborhood and letting $f := (f_1, \ldots, f_n)\colon U \to \mathbb{A}^n$ we have $(x, f)\colon \underline{U} \to \mathfrak{X} \times \underline{\mathbb{A}}^n$ étale.
- For $Y \subset \mathbb{A}^n$ which is étale (over Spec($\mathbb{Z}$)) we have $\underline{f^{-1}(Y)} \to \mathfrak{X}$ étale, and if Y is chosen suitably then we can set $U_w := f^{-1}(Y)$ and apply Lemma 5.25 (to the cover $\underline{U} \to \mathfrak{X}$) to conclude that $\mathfrak{X}$ is a DM stack.

The last step is the "slice" step. In this step we see that the morphism x, restricted a general enough slice (locally closed subscheme of codimension n) of U, will be étale. However it is not possible in general to arrange for the slice to pass *through* the given point w. There can be bad points through which no slice is étale over $\mathfrak{X}$; for an example of this, see Exercise 5.7. The content of Lemma 5.25 is that it is enough for there to exist a point $p \in R$ such that $s(p)$ lies on a good slice and $t(p) = w$.

We recall that the sheaf of relative differentials of a morphism of schemes $X \to Y$ is $\Omega_{Y/X} := \mathcal{N}_{Y/Y\times_X Y}$, the conormal sheaf to the relative diagonal (cf. the Glossary). We have $x\colon \underline{U} \to \mathfrak{X}$, a morphism where the target is a stack, rather than a scheme, but since $\underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}$ we have a relative diagonal $e\colon U \to R$ that is a morphism of schemes. So we define

$$\Omega_x := \mathcal{N}_{U/R} = \mathcal{N}_e,$$

the conormal sheaf to the identity morphism of the groupoid scheme. The composite

$$U \to R \to U \times U$$

gives rise to a pullback map $\mathcal{N}_{U/U\times U} \to \mathcal{N}_{U/R}$, and this we take as our the morphism

$$\varphi\colon \Omega_U \to \Omega_x.$$

There is an important compatibility after pullback by s. By the cartesian diagram

$$\begin{array}{ccccc} R & \xrightarrow{(e\circ s, 1_R)} & R\ {}_t\!\times_s R & \xrightarrow{\mathrm{pr}_2} & R \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} \\ U & \xrightarrow{e} & R & \xrightarrow{t} & U \end{array}$$

we have

$$s^*\Omega_x \cong \mathcal{N}_{R/R\ {}_t\!\times_s R} \cong \Omega_t \tag{10}$$

where the latter isomorphism uses the identification of $R\ {}_t\!\times_s R$ with the fiber product of t with itself, by (m, pr_2). Now we claim that this fits into a commutative diagram

$$\begin{array}{ccc} s^*\Omega_U & \xrightarrow{s^*\varphi} & s^*\Omega_x \\ \downarrow & & \downarrow{\scriptstyle \sim} \\ \Omega_R & \longrightarrow & \Omega_t \end{array} \tag{11}$$

where the left-hand and bottom morphisms are the pullback morphism on differentials by the morphism s and the morphism from differentials on R to relative differentials by t, respectively. Each of these sheaves of differentials is a conormal sheaf:

$$\begin{array}{ccc} s^*\mathcal{N}_{U/U\times U} & \longrightarrow & s^*\mathcal{N}_{U/R} \\ \big\downarrow & & \big\downarrow \\ \mathcal{N}_{R/R\times R} & \longrightarrow & \mathcal{N}_{R/R\ {}_t\times_s R} \end{array}$$

This diagram commutes because formation of the conormal sheaf is functorial ([**EGA** IV.16.2.1]) and we have the following commutative diagram:

$$\begin{array}{ccccc} R & \xrightarrow{(e\circ s,1_R)} & R\ {}_t\times_s R & \xrightarrow{(m,\mathrm{pr}_2)} & R\times R \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle s\times s} \\ U & \xrightarrow{\ \ e\ \ } & R & \xrightarrow{(s,t)} & U\times U \end{array}$$

So far we have used only hypotheses (1) and (2). By commutativity of (11) and the fact that s is smooth and surjective, the surjectivity of φ is equivalent to surjectivity of the composite morphism

$$s^*\Omega_U \to \Omega_t. \tag{12}$$

The first fundamental exact sequence of differentials (cf. the Glossary) of the sequence of morphisms

$$R \xrightarrow{(s,t)} U\times U \xrightarrow{\mathrm{pr}_2} U$$

is

$$s^*\Omega_U \to \Omega_t \to \Omega_{R/U\times U} \to 0$$

(where we are identifying Ω_{pr_2} with $\mathrm{pr}_1^*\,\Omega_U$ and using functoriality of the induced morphisms on sheaves of differentials). Hence surjectivity of (12) is equivalent to the vanishing of $\Omega_{R/U\times U}$. And this is precisely condition (3′). So, we have established that the morphism φ is surjective.

Let $w\in U$ be a closed point. We have $w\in U_n$ for some $n=n(w)$, which is the rank of Ω_x at w. Since φ is surjective, there exists an open neighborhood of w in U_n, and regular functions $f_1, \ldots, f_n$ such that $df_1, \ldots, df_n$ generate Ω_x on this neighborhood. To maintain simplicity of notation, let us replace U momentarily by this neighborhood (and replace R by the open subset of points which map by both s and t into this neighborhood, so $R \rightrightarrows U$ is again a groupoid scheme; cf. Example 3.15). So $f_1, \ldots, f_n\in\Gamma(U,\mathcal{O}_U)$, and the morphism

$$f := (f_1,\ldots,f_n)\colon U\to\mathbb{A}^n$$

is such that the composite

$$\mathcal{O}_U^{\oplus n} \cong f^*\Omega_{\mathbb{A}^n} \to \Omega_U \to \Omega_x \tag{13}$$

is an isomorphism.

There is a 2-cartesian diagram

(14)
$$\begin{array}{ccc} \underline{R} & \xrightarrow{(t,f\circ s)} & \underline{U}\times\underline{\mathbb{A}}^n \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle x\times 1_{\mathbb{A}^n}} \\ \underline{U} & \xrightarrow{(x,t)} & \mathfrak{X}\times\underline{\mathbb{A}}^n \end{array}$$

So by Proposition 5.9 and Remark 5.24, the morphism $(x,t)\colon \underline{U}\to\mathfrak{X}\times\underline{\mathbb{A}}^n$ is étale if and only if the morphism
$$(t,f\circ s)\colon R\to U\times\mathbb{A}^n$$
is étale. By [**EGA** IV.17.11.2], given two schemes smooth over some base scheme, a given morphism (over the base schemes) is étale if and only if the induced morphism on relative differentials is an isomorphism. Viewing R and $U\times\mathbb{A}^n$ as schemes over U, this induced morphism is
$$\mathcal{O}_R^{\oplus n}\cong(t,f\circ s)^*\Omega_{U\times\mathbb{A}^n/U}\to\Omega_t.$$
This composite factors as
$$\mathcal{O}_R^{\oplus n}\cong(f\circ s)^*\Omega_{\mathbb{A}^n}\to s^*\Omega_U\to\Omega_t$$
where the last morphism is the morphism (12), which we have seen factors, further, through $s^*\Omega_x$ in (11). So the induced morphism on differentials is s^* of the isomorphism (13) followed by the isomorphism (10), and the second step has been accomplished.

For any locally closed subscheme $Y\subset\mathbb{A}^n=\mathbb{A}^n_{\mathrm{Spec}(\mathbb{Z})}$ we have a diagram

$$\begin{array}{ccccc} \underline{f^{-1}(Y)} & \longrightarrow & \mathfrak{X}\times\underline{Y} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow & & \\ \underline{U} & \longrightarrow & \mathfrak{X}\times\underline{\mathbb{A}}^n & & \end{array}$$

with 2-cartesian square, where the bottom horizontal map is étale, and hence the horizontal map above it is also étale. If Y is étale over $\mathrm{Spec}(\mathbb{Z})$, then the projection morphism $\mathfrak{X}\times\underline{Y}\to\mathfrak{X}$ is étale, so the composite morphism of the top row is an étale morphism. We will restrict attention to $Y\subset\mathbb{A}^n$ that are étale over $\mathrm{Spec}(\mathbb{Z})$. To complete the third step, and hence the entire proof (by applying Lemma 5.25 with $U_w=f^{-1}(Y)$), we need to show that étale Y can be chosen, so that $(f\circ s)^{-1}(Y)$ has nontrivial intersection with the fiber $t^{-1}(w)$. The left-hand morphism in the composite

(15)
$$t^{-1}(w)\to\mathbb{A}^n_{k(w)}\to\mathbb{A}^n$$

is obtained by base change of the top morphism in (14) by $\{w\}\times\mathbb{A}^n\to U\times\mathbb{A}^n$, hence is étale, and in particular is open. Here $k(w)$ denotes the residue field of w. There are now two cases to consider. If $k(w)$ has characteristic 0, then there exists a rational point $(z_1,\ldots,z_n)\in\mathbb{A}^n_{\mathbb{Q}}$ over which the composite (15) has nonempty fiber. If $k(w)$ has characteristic $p>0$ then there exists a point of $t^{-1}(w)$ whose residue field is a finite extension of the prime field $\mathbb{F}_p$, and this point has image which is a closed point $z\in\mathbb{A}^n$. In either case, there exists étale $Y\subset\mathbb{A}^n$ containing z. In the characteristic

0 case, Y can be defined by inverting a suitable positive integer N and equating the ith generator of the coordinate ring of $\mathbb{A}^n$ with z_i for each i. In characteristic p a regular system of parameters of $\mathcal{O}_{\mathbb{A}^n_{\mathbb{F}_p},z}$ can be lifted to characteristic 0 and Y can be taken to be a suitable neighborhood of z in the scheme defined by these equations; cf. [**EGA** IV.17.16.3(i)]. □

For the transformation groupoid of a group action, the most natural sort of condition to consider is a condition on the (geometric) stabilizers. The next result says that to verify condition (3′) of Theorem 5.26 it suffices to check that the stabilizer of a groupoid scheme is unramified.

PROPOSITION 5.28. *Let $s, t\colon R \rightrightarrows U$ be a smooth groupoid scheme with quasi-compact separated relative diagonal. Define the stabilizer of $R \rightrightarrows U$ to be top morphism in the cartesian diagram*

(16)
$$\begin{array}{ccc} S & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle \Delta_U} \\ R & \xrightarrow{(s,t)} & U \times U \end{array}$$

If the stabilizer has finite reduced geometric fibers then $R \to U \times U$ is formally unramified.

So, if an algebraic group G (or, more generally, a group scheme, smooth and separated over the base scheme) acts on a quasi-separated scheme X with finite reduced geometric stabilizers, then $[X/G]$ is a DM stack.

PROOF. Let Ω be an algebraically closed field, and let x, $y \in U(\Omega)$. Let us denote by S_x the fiber of the stabilizer over x. Set $R_{x,y} := (s,t)^{-1}(x,y)$; we need to show that $R_{x,y}$ is reduced and finite over $\mathrm{Spec}(\Omega)$. Assume that $R_{x,y}$ is nonempty, and choose $z \in R(\Omega)$ satisfying $s(z) = x$ and $t(z) = y$. Now $m(-, i(z))$ is a morphism

(17)
$$R_{x,y} \to S_x.$$

Multiplication with z, by the groupoid scheme axioms, gives rise to a morphism in the other direction, establishing that (17) is an isomorphism. Under the hypothesis, S_x is finite and reduced, hence so is $R_{x,y}$. □

EXAMPLE 5.29. Let V be a representation of $\mathbb{G}_m$, over a field k. If none of the weights of the $\mathbb{G}_m$ action divide the characteristic of k then $[V \smallsetminus \{0\}/\mathbb{G}_m]$ is DM stack. In Chapter 10 we will study DM stacks of the form $[\mathbb{A}^2 \smallsetminus \{0\}/\mathbb{G}_m]$ in some detail.

EXERCISE 5.7. Consider the 2-dimensional representation of $\mathbb{G}_m$ with weights 1 and 1 over the field $k = \mathbb{F}_p(t)$, where p is any prime number.

(i) We have $[\mathbb{A}^2_k \smallsetminus \{0\}/\mathbb{G}_m] \cong \underline{\mathbb{P}}^1_k$.

(ii) If we call $C \subset \mathbb{A}^2_k \smallsetminus \{0\}$ a *good slice* if $\underline{C} \to [\mathbb{A}^2_k \smallsetminus \{0\}/\mathbb{G}_m]$ is étale, then necessary and sufficient conditions for C to be a good slice are that C is smooth (over k) of pure dimension 1 and the tangent line to C at every geometric point does not pass through the origin of $\mathbb{A}^2$.

(iii) There exist points in $\mathbb{A}^2_k \smallsetminus \{0\}$ which are not contained in any good slice.

EXAMPLE 5.30. The stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are DM stacks. This will be shown in Chapter 7 by showing that they are isomorphic to quotient stacks, for an action of a projective linear group on a locus in a Hilbert scheme, with finite reduced geometric stabilizers.

EXAMPLE 5.31. The stack $\mathcal{A}_g$ of principally polarized g-dimensional abelian varieties is a DM stack over $\operatorname{Spec}\mathbb{Z}$. These stacks admit toroidal compactifications $\overline{\mathcal{A}}_g$ which are also DM stacks over $\operatorname{Spec}\mathbb{Z}$. For a reference see the book by Faltings and Chai [**25**].

PROPOSITION 5.32. (a) *If $\mathfrak{X} \to \mathfrak{Y}$ is a representable quasi-separated morphism and $\mathfrak{Y}$ is a DM stack, then $\mathfrak{X}$ is a DM stack.*

(b) *If $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ are DM stacks then any fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is a DM stack.*

(c) *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ are morphisms of DM stacks, and if $g \circ f$ is representable, then f is representable.*

PROOF. For (a), we know that $\mathfrak{X}$ has representable diagonal by Proposition 5.19(b). The proof of that statement uses the factorization of the diagonal of $\mathfrak{X}$, in (7), as the relative diagonal of $\mathfrak{X} \to \mathfrak{Y}$ followed by a morphism to $\mathfrak{X} \times \mathfrak{X}$. The latter morphism is obtained by base change from the diagonal of $\mathfrak{Y}$ (Lemma 5.18), and hence is quasi-compact and separated. The former morphism, when composed with $\mathrm{pr}_2\colon \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X}$, yields $1_{\mathfrak{X}}$. And pr_2 is obtained from base change by $\mathfrak{X} \to \mathfrak{Y}$, hence is representable and quasi-separated. Now [**EGA** I.5.5.1(v)] and [**EGA** IV.1.2.4] tell us that when $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of schemes with g quasi-separated and $g \circ f$ separated and quasi-compact, then f is separated and quasi-compact. Since $1_{\mathfrak{X}}$ is separated and quasi-compact, it follows that $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is separated and quasi-compact. So $\mathfrak{X}$ has representable, separated, quasi-compact diagonal. If $\underline{V} \to \mathfrak{Y}$ is an étale surjective morphism, and U is a scheme with $\underline{U} \cong \mathfrak{X} \times_{\mathfrak{Y}} \underline{V}$, then we have étale surjective $\underline{U} \to \mathfrak{X}$, and so $\mathfrak{X}$ is a DM stack.

Using the isomorphism $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \cong \mathfrak{Z} \times_{\mathfrak{Z} \times \mathfrak{Z}} (\mathfrak{X} \times \mathfrak{Y})$, the special case $\mathfrak{Z} = \underline{\Lambda}$ of (b), combined with (a), implies the general case of (b). So we are reduced to showing that $\mathfrak{X} \times \mathfrak{Y}$ is a DM stack. The stack $\mathfrak{X} \times \mathfrak{Y}$ has representable diagonal since its diagonal factors as $\mathfrak{X}\times\mathfrak{Y} \to \mathfrak{X}\times\mathfrak{X}\times\mathfrak{Y}\times\mathfrak{Y} \to \mathfrak{X}\times\mathfrak{Y}\times\mathfrak{X}\times\mathfrak{Y}$, a product of representable morphisms (which is representable by Proposition 5.8(iv)) composed with an isomorphism. Since a product of separated morphisms is separated and a product of quasi-compact morphisms is quasi-compact ([**EGA** I.5.5.1](iii) and [**EGA** I.6.6.4](iv)), the diagonal of $\mathfrak{X} \times \mathfrak{Y}$ is quasi-compact and separated. Since a product of étale surjective morphisms is étale surjective, we have $\underline{U \times V} \to \mathfrak{X} \times \mathfrak{Y}$, where U and V denote étale covers of $\mathfrak{X}$ and of $\mathfrak{Y}$, and (b) is established. (We get these assertions about products of representable morphisms by following the proof of Proposition 5.8(iv), e.g., $(\underline{U} \times \underline{V}) \times_{\mathfrak{X}\times\mathfrak{Y}} \underline{T} \cong (\underline{U} \times_{\mathfrak{X}} \underline{T}) \times_{\underline{T}} (\underline{V} \times_{\mathfrak{Y}} \underline{T})$.)

Lastly, (c) follows directly from Proposition 5.19(c). □

REMARK 5.33. In the proof of Proposition 5.32, we could observe, moreover, that the morphism $1_{\mathfrak{X}}$ is locally of finite type, hence by [**EGA** I.6.6.6] $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is also locally of finite type. We have already seen that it is quasi-compact and separated. So, the relative diagonal $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ of an arbitrary morphism of DM stacks $\mathfrak{X} \to \mathfrak{Y}$ is separated and of finite type.

Answers to Exercises

5.1. The property of being a regular embedding is local for the étale topology but not invariant under base change. Being projective is invariant under base change but not even Zariski local.

5.2. This follows from $(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}') \times_{\mathfrak{Y}'} \underline{T} \cong \mathfrak{X} \times_{\mathfrak{Y}} \underline{T}$.

5.3. Let S be a scheme, and $h\colon S \to T$ a morphism. Then a G-equivariant isomorphism $S \times G \to S \times G$ is given by $(s, g) \mapsto (s, \alpha(s)g)$ for a unique morphism $\alpha\colon S \to G$. To be compatible with the morphisms to X is the condition $f(h(s)) \cdot g = g(h(s)) \cdot (\alpha(s)g)$. Hence $\mathcal{I}som_{[X/G]}(x, y)$ is represented by the scheme $(T \times G) \times_{X \times X} X$ where the morphism $T \times G \to X \times X$ sends (t, g) to $(f(t), g(t) \cdot \alpha(s))$ and where the morphism $X \to X \times X$ is the diagonal.

5.4. Given any scheme U and surjective morphism $f\colon \underline{U} \to BG$, corresponding to a G-torsor $E \to U$, we have a fiber diagram

$$\begin{array}{ccc} \underline{E} & \xrightarrow{f'} & \Lambda \\ \downarrow & & \downarrow \\ \underline{U} & \xrightarrow{f} & BG \end{array}$$

The morphism f' has positive-dimensional fibers, hence cannot be étale. So f is not an étale morphism, and Axiom (2) for BG fails.

5.5. We get $\mathfrak{X} \cong [R \rightrightarrows X]$ where R is an inner form of S_3 over X, specifically $R = X \coprod X \coprod Y \coprod Y$.

5.6. The stack $\mathfrak{X}$ has representable, separated, quasi-compact diagonal by Proposition 5.21. Suppose $\underline{V} \to \mathfrak{X}$ is étale; we show that the image of $\underline{V} \times_{\mathfrak{X}} \underline{U} \to \underline{U}$ must not contain u, hence $\underline{V} \to \mathfrak{X}$ is not surjective. Replacing V by an étale cover we may assume that $\underline{V} \to \mathfrak{X}$ factors up to 2-isomorphism through $\underline{U}$ (since by definition objects of $\mathfrak{X}$ are étale locally trivial $(R \rightrightarrows U)$-torsors). Consider the 2-cartesian diagram

$$\begin{array}{ccccc} \underline{V} \times_{\mathfrak{X}} \underline{U} & \longrightarrow & \underline{R} & \xrightarrow{t} & \underline{U} \\ \downarrow & & \downarrow{\scriptstyle s} & & \downarrow \\ \underline{V} & \longrightarrow & \underline{U} & \longrightarrow & \mathfrak{X} \end{array}$$

The composite top map is étale, hence $\underline{V} \times_{\mathfrak{X}} \underline{U}$ is representable by a scheme that is smooth over the complex numbers. Now V must be locally of finite type (over the complex numbers), since the morphism $V \to U$, after base change by s, becomes a morphism that is locally of finite type. By [**EGA** IV.17.7.7] a scheme that admits an fppf cover by a smooth scheme must be smooth, so V must be smooth. It follows that

$V \to U$ is flat, hence so is $\underline{V} \times_{\mathfrak{X}} \underline{U} \to \underline{R}$, and so the image of the latter morphism must be contained in the smooth locus of R. Hence it does not contain the (unique) point of R lying above x.

5.7. As discussed after the statement of Theorem 5.26 we have $[\mathbb{A}^2_k \smallsetminus \{0\}/\mathbb{G}_m] \cong [(\mathbb{A}^2_k \smallsetminus \{0\}) \times \mathbb{G}_m \rightrightarrows \mathbb{A}^2_k]$. The obvious map $\mathbb{A}^2_k \smallsetminus \{0\} \to \mathbb{P}^1_k$ is a $\mathbb{G}_m$-torsor, i.e., $(\mathbb{A}^2 \smallsetminus \{0\}) \times_{\mathbb{P}^1} (\mathbb{A}^2 \smallsetminus \{0\}) \cong (\mathbb{A}^2 \smallsetminus \{0\}) \times \mathbb{G}_m$, hence $[(\mathbb{A}^2_k \smallsetminus \{0\}) \times \mathbb{G}_m \rightrightarrows \mathbb{A}^2_k] \cong \mathbb{P}^1$ by Remark 5.24. Hence C is good if and only if $C \to \mathbb{P}^1_k$ is étale, and this is equivalent to the stated conditions. If C contains the point defined by $x^2 - t = y = 0$, then we see by computing with a local defining equation that C has horizontal tangent line through the point $(\sqrt{t}, 0)$.