CHAPTER 4

Stacks and Stackification

In this chapter we will endow our base category $\mathcal{S}$ with an additional structure, a *Grothendieck topology.* Using Grothendieck topologies it makes sense to speak of sheaves over a *category.* To a topological space, for instance, one associates the category of all open sets with inclusion maps. A sheaf over a topological space is given in terms of this category. Reflecting upon the definition of sheaves, one discovers that the essential notion needed to write down what a sheaf is, is that of a *covering family.* For the category associated with a topological space, these are just families of inclusions that cover the image.

Several of our examples drawn from algebraic geometry have illustrated the shortcomings of the usual Zariski topology. For many groups G (e.g., finite groups), G-torsors are almost never locally trivial for the Zariski topology. When we encountered modular families of elliptic curves and noted the property that every family can be analytically locally obtained from these by pullback, it was shown by example that this statement is no longer true in the algebraic setting if we use the Zariski topology. What works in both of these cases is to interpret "locally" to mean locally for the *étale topology*, which is a Grothendieck topology. For the formal definition of a Grothendieck topology see the Glossary.

A category endowed with a Grothendieck topology is called a *site.* Thus it makes sense to speak of sheaves on a site. We use the topology on $\mathcal{S}$ to define what it means for a CFG over $\mathcal{S}$ to be a *stack.* In fact, there will be two stack axioms. If a CFG satisfies only the first it is called a *prestack.* If the CFG is that associated with a presheaf as in Example 2.4, then the stack axioms reduce to the sheaf axioms. In this way, stacks appear as generalizations of sheaves.

In our "key case", where $\mathcal{S}$ is the category of schemes over the base scheme Λ, we put the étale topology on $\mathcal{S}$. In this topology a family of maps $\{U_i \to S\}$ is a covering family if all structure maps $U_i \to S$ are étale and $\coprod_i U_i \to S$ is surjective.

1. The stack axioms

Definition 4.1. A **site** is a category together with a Grothendieck topology. When the choice of Grothendieck topology on a category $\mathcal{S}$ is understood, we will often omit explicit mention of it and refer, e.g., to a CFG over a site $\mathcal{S}$.

Let $\mathfrak{X} \to \mathcal{S}$ be a CFG over a site $\mathcal{S}$. The key case is that of $\mathcal{S}$ being the category of schemes over Λ with the étale topology, where Λ a fixed base scheme. The reader may consider only this case at a first reading. Another good example to keep in mind is the category of open subsets of a topological space, and inclusions of open subsets.

Let x and y be objects of $\mathfrak{X}$ over the same object S of $\mathcal{S}$. Recall, if T is any scheme over S, with structural map $f\colon T \to S$, there are pullbacks $f^*(x)$ and $f^*(y)$, each defined up to canonical isomorphism. We define a presheaf $\mathcal{I}som_{\mathfrak{X}}(x,y)$ by setting

$$\mathcal{I}som_{\mathfrak{X}}(x,y)(T) = \{\text{isomorphisms from } f^*(x) \text{ to } f^*(y) \text{ in } \mathfrak{X}_T\}.$$

Because this is defined for T equipped with a structural map to S, it is a presheaf on the category $\underline{S}$. (More common notation, in this context, would be $\mathcal{S}/S$, the *slice category* of schemes over S.) The definition appears to depend on choices of pullbacks $f^*(x)$ and $f^*(y)$, but if $\bar{x}$ and $\bar{y}$ denote other choices of pullbacks, then using the canonical isomorphisms $\bar{x} \cong f^*(x)$ and $\bar{y} \cong f^*(y)$ we can canonically identify the set of isomorphisms $\bar{x} \to \bar{y}$ with the set of isomorphisms $f^*(x) \to f^*(y)$. To be a presheaf means there are restriction maps: given $g\colon T' \to S$ and a morphism $h\colon T' \to T$ such that $g = f \circ h$, then there is a unique morphism $\psi\colon g^*(x) \to g^*(y)$ in $\mathfrak{X}_U$ whose composite with $g^*(y) \to f^*(y)$ is equal to the composite $g^*(x) \to f^*(x) \xrightarrow{\varphi} f^*(y)$ for some given $\varphi \in \mathcal{I}som_{\mathfrak{X}}(x,y)(T)$. Then $h^*(\varphi)$ is this morphism ψ.

The category $\underline{S}$ inherits a Grothendieck topology from $\mathcal{S}$, in which a set of morphisms in $\underline{S}$ is deemed a covering family if the collection of underlying morphisms of schemes in $\mathcal{S}$ is a covering family. The first of the stack axioms is that this presheaf is a sheaf for this inherited topology.

Axiom 1. If $\{T_\alpha \to T\}$ is a covering family in the category of schemes over S, then

$$\mathcal{I}som_{\mathfrak{X}}(x,y)(T) \to \prod_\alpha \mathcal{I}som_{\mathfrak{X}}(x,y)(T_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{I}som_{\mathfrak{X}}(x,y)(T_\alpha \times_T T_\beta)$$

is an exact sequence of sets.[1]

Definition 4.2. A CFG $\mathfrak{X}$ which satisfies Axiom 1 (for every object S of $\mathcal{S}$ and pair of objects x and y of $\mathfrak{X}$ over S) is called a **prestack**.

Let us verify Axiom 1 in some of our examples. For $\mathcal{M}_g$, given two families $\pi\colon C \to S$ and $\pi'\colon C' \to S$, and given $T \to S$, then $\mathcal{I}som_{\mathcal{M}_g}(\pi,\pi')(T)$ is the set of isomorphisms $T \times_S C \to T \times_S C'$ over T. Let $\{T_\alpha \to T\}$ be a covering family. Let $\{T_\alpha \times_S C \to T_\alpha \times_S C'\}$ be isomorphisms. The condition to pull back by either projection to the same element of $\mathcal{I}som_{\mathcal{M}_g}(\pi,\pi')(T_\alpha \times_T T_\beta)$ for every α and β is equivalent to the equality of the two composite morphisms, involving the two projections:

$$\coprod_{\alpha,\beta} T_\alpha \times_T T_\beta \times_S C \rightrightarrows \coprod_\alpha T_\alpha \times_S C \to T \times_S C'.$$

By descent for morphisms to a scheme (Proposition A.13), this condition implies there is a unique morphism $T \times_S C \to T \times_S C'$ whose composite with $\coprod T_\alpha \times_S C \to T \times_S C$ is that indicated above. This morphism is an isomorphism because it becomes an isomorphism after faithfully flat base change (by [**EGA** IV.2.7.1]).

To verify that BG is a prestack, we use descent for morphisms as above to prove the existence of an isomorphism given one locally. It remains to see that it is G-equivariant,

[1] A sequence of sets $A \to B \rightrightarrows C$ is exact if A is mapped bijectively onto the set of elements in B which have the same image in C by the two maps from B to C.

but this amounts to checking equalities of morphisms, and again we use descent for morphisms (actually just the uniqueness part of descent for morphisms). The same argument applies for $\mathcal{M}_{g,n}$: we use descent for morphisms to produce the isomorphism, and then the uniqueness assertion of descent for morphisms to check compatibilities of the sections. Similar arguments apply for $[X/G]$, $\mathcal{V}_n$ (vector bundles), $\mathcal{C}_n$ (n-sheeted coverings) and the other variants of moduli stacks $\overline{\mathcal{M}}_g$, $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

The same kind of argument also applies to the CFG $[R \rightrightarrows U]^{\mathrm{pre}}$ obtained from a groupoid scheme $R \rightrightarrows U$. In this CFG, we recall, a morphism is given by a map to R. So, if $R \rightrightarrows U$ is any groupoid scheme, the CFG $[R \rightrightarrows U]^{\mathrm{pre}}$ is a prestack.

2. Stacks

A prestack is a *stack* if it satisfies a descent-type hypothesis, to the effect that an object can be constructed locally by gluing. We make use of projection maps $p_1\colon T_\alpha \times_T T_\beta \to T_\alpha$ and $p_2\colon T_\alpha \times_T T_\beta \to T_\beta$, or for $T' \to T$, projection maps $p_1, p_2\colon T'' \to T'$ where $T'' = T' \times_T T'$.

Axiom 2. If $\{T_\alpha \to T\}$ is a covering family, then given any collection of objects t_α over T_α and isomorphisms $\varphi_{\alpha\beta}\colon p_1^* t_\alpha \to p_2^* t_\beta$ over $T_\alpha \times_T T_\beta$ satisfying the cocycle condition, there is an object x over T and for each α, an isomorphism $\lambda_\alpha\colon x_\alpha \to t_\alpha$, where x_α denotes a pullback to T_α. These isomorphisms are required to satisfy the natural compatibility condition on $T_\alpha \times_T T_\beta$.

The cocycle condition states that, with projections $p_{12}\colon T_\alpha \times_T T_\beta \times_T T_\gamma \to T_\alpha \times_T T_\beta$, etc., the diagram

$$\begin{tikzcd}
p_{12}^* p_1^* t_\alpha \arrow[r, "p_{12}^*\varphi_{\alpha\beta}"] \arrow[d, equal] & p_{12}^* p_2^* t_\beta \arrow[r, equal] & p_{23}^* p_1^* t_\beta \arrow[d, "p_{23}^*\varphi_{\beta\gamma}"] \\
p_{13}^* p_1^* t_\alpha \arrow[r, "p_{13}^*\varphi_{\alpha\gamma}"'] & p_{13}^* p_2^* t_\gamma \arrow[r, equal] & p_{23}^* p_2^* t_\gamma
\end{tikzcd}$$

commutes, where the equal signs denote canonical isomorphisms of pullbacks. The natural compatibility condition on $T_\alpha \times_T T_\beta$ is the commutativity of the following diagram

$$\begin{tikzcd}
p_1^* x_\alpha \arrow[r, "p_1^*\lambda_\alpha"] \arrow[d, equal] & p_1^* t_\alpha \arrow[d, "\varphi_{\alpha\beta}"] \\
p_2^* x_\beta \arrow[r, "p_2^*\lambda_\beta"'] & p_2^* t_\beta
\end{tikzcd}$$

(again involving a canonical isomorphism of pullbacks denoted with an equal sign).

Definition 4.3. A CFG $\mathfrak{X}$ is a **stack** if it satisfies both Axiom 1 and Axiom 2.

In Axiom 2, the tuple of objects x_α, together with isomorphisms $\varphi_{\alpha\beta}$ satisfying the cocycle condition, is called a **descent datum**. If the conclusion of Axiom 2 holds, we say the descent datum is **effective**. The condition in Axiom 2 can alternatively be expressed by writing T' for the disjoint union of all the T_α, and then speaking of a single object over T' and an isomorphism of its pullbacks to $T'' = T' \times_T T'$. In practice this gives an equivalent formulation of Axiom 2. But this involves two subtleties. First, the disjoint union of the T_α is not required to exist in the category $\mathcal{S}$. Second, if we restate Axiom 2 using only one-element covering families, then there will be no provision requiring a choice of objects t_i over T_i, for some collection of i, to determine an object over $\coprod T_i$. The category of schemes has arbitrary disjoint unions. The second issue is avoided when the following hypothesis is satisfied.

Hypothesis. We suppose $\mathcal{S}$ is the category of schemes over Λ with the étale topology, and $\mathfrak{X}$ is a CFG which satisfies: for any collection of schemes S_α, if we set $S = \coprod_\alpha S_\alpha$, then some (or equivalently, any) choice of change of base functors $\mathfrak{X}_S \to \mathfrak{X}_{S_\alpha}$ determines an equivalence of categories

$$\mathfrak{X}_S \to \prod_\alpha \mathfrak{X}_{S_\alpha}.$$

(The product category, on the right, is the category whose objects are tuples of objects in $\mathfrak{X}_{S_\alpha}$ for each α, and whose morphisms are tuples of morphisms.)

Assuming the Hypothesis, Axiom 2 is equivalent to:

Axiom 2′. If $f\colon T' \to T$ is a covering map (meaning that $\{f\}$ is a one-element covering family), and x' is any object over T', with isomorphism $\varphi\colon p_1^*x' \to p_2^*x'$ satisfying the cocycle condition $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$, then there exists an object x over T and an isomorphism $f^*x \to x'$ over T' such that $p_2^*\lambda = \varphi \circ p_1^*\lambda$.

The Hypothesis is satisfied by all of CFGs over schemes that we have seen as examples. An advantage of Axiom 2′ is that it nicely parallels many assertions from the theory of descent (Appendix A). For instance, when G is an affine group scheme over the base scheme Λ, then Axioms 1 and 2′ are implied by (a) and (b), respectively, of Corollary A.16. This gives us the first of several examples of stacks that we now list:

(1) BG is a stack, for any affine group scheme G over Λ.
(2) $\underline{X}$ is a stack for any scheme X: Axioms 1 and 2 follow from descent for morphisms to X.
(3) More generally, if $\mathfrak{X} = \underline{h}$ is the CFG associated to a presheaf h on $\mathcal{S}$, then the stack axioms for $\mathfrak{X}$ are equivalent to the sheaf axioms.
(4) Combining the first two examples, the CFG $[X/G]$ is a stack, for any affine group scheme G acting on a scheme X, respectively.
(5) The following are stacks: $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ for $g \geq 2$; $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ for $2g+n \geq 3$; $\overline{\mathcal{M}}_{g,n}(X,\beta)$. To show these are stacks, use Proposition A.18 to verify Axiom 2′, applied to the relative dualizing sheaf of a family of (stable or smooth) curves; the relative dualizing sheaf twisted by the sections; or twisted by the pullback of an ample line bundle on the projective variety X.
(6) The CFG $\mathcal{V}_n$ is a stack by Proposition A.11. The CFG $\mathcal{S}_n$ is a stack by Proposition A.12.

The hypothesis that G is an affine group scheme is a convenient one because it is satisfied for the most common linear algebraic groups (GL_n, PGL_n, etc.), as well as for finite groups. However the assertions can be generalized to the case G quasi-affine by appealing to the more general descent result of Proposition A.17.

Exercise 4.1. Which of the following CFGs are prestacks? Which are stacks? (a) the CFG of families of (smooth) genus 0 curves. (b) The category of finite flat covers $E \to S$ of degree d (for some integer d). (c) The CFG associated with the presheaf whose sections on S are the isomorphism classes of families of elliptic curves over S. (d) The category of projectivized vector bundles $\mathbb{P}(V) \to S$.

If $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ are stacks, and $\mathfrak{X} \to \mathfrak{Z}$ and $\mathfrak{Y} \to \mathfrak{Z}$ are morphisms, then the fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is a stack. It is a matter of routine verification of axioms to show this. E.g., a descent datum for $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ consists of a descent datum for $\mathfrak{X}$, a descent datum for $\mathfrak{Y}$, and compatible isomorphisms in $\mathfrak{Z}$; by Axiom 2 for $\mathfrak{X}$ and $\mathfrak{Y}$ we produce objects and by Axiom 1 for $\mathfrak{Z}$ we produce an isomorphism, which taken together show that the descent datum for $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is effective.

Remark 4.4. The stack axioms for a CFG on a general site can be stated in a way that avoids any reference to the presheaf $\mathcal{I}som_{\mathfrak{X}}(x, y)$. It will be convenient to have these reformulations below, in §4.4. For Axiom 1, the CFG axioms let us identify isomorphisms $f^*(x) \to f^*(y)$ in $\mathfrak{X}_T$ with morphisms $f^*(x) \to y$ over $f\colon T \to S$. Moreover the axiom applies in case x is an object over T rather than over S (by applying the axiom as stated to the objects x and $f^*(y)$ over T). So the axiom is equivalent to:

For any $f\colon T \to S$, objects y over S and t over T, covering family $\{T_\alpha \to T\}$ and morphisms $t_\alpha \to t$ over $T_\alpha \to T$, let $t_{\alpha\beta} \to t_\alpha$ and $t_{\alpha\beta} \to t_\beta$ be morphisms over the respective projections from $T_{\alpha\beta} := T_\alpha \times_T T_\beta$ such that the composite morphisms to t are equal. Then, composition with $t_\alpha \to t$ induces a bijection between morphisms $t \to y$ over f and tuples $(t_\alpha \to y)_\alpha$ over $T_\alpha \to S$ such that the diagram

$$\begin{tikzcd} t_{\alpha\beta} \arrow[r] \arrow[d] & t_\beta \arrow[d] \\ t_\alpha \arrow[r] & y \end{tikzcd}$$

commutes, for every α and β.

There is also a restatement of Axiom 2. Pullbacks in a CFG are only determined up to isomorphism, so there is no loss of generality in assuming every isomorphism $\varphi_{\alpha\beta}$ in the axiom to be the identity. This means objects $t_{\alpha\beta}$ are given, each with a morphism to t_α identifying $t_{\alpha\beta}$ with $p_1^* t_\alpha$ and a morphism to t_β identifying $t_{\alpha\beta}$ also with $p_2^* t_\beta$. We introduce $T_{\alpha\beta\gamma} := T_\alpha \times_T T_\beta \times_T T_\gamma$. If $t_{\alpha\beta\gamma} \to t_{\alpha\beta}$ is a morphism over $p_{12}\colon T_{\alpha\beta\gamma} \to T_{\alpha\beta}$, then the CFG axioms dictate a unique $t_{\alpha\beta\gamma} \to t_{\beta\gamma}$ over p_{23} making a commutative square with t_β. Axiom 2, restated, is that in the diagram

$$\begin{tikzcd} & t_{\alpha\beta\gamma} \arrow[dl] \arrow[d, bend right] \arrow[d, bend left] \arrow[dr] & \\ t_{\alpha\beta} \arrow[d] \arrow[dr] & t_{\alpha\gamma} \arrow[dl] \arrow[dr] & t_{\beta\gamma} \arrow[dl] \arrow[d] \\ t_\alpha & t_\beta & t_\gamma \end{tikzcd}$$

if the two curved arrows, defined to be the unique morphisms over p_{13} making the left-hand square resp. right-hand square commutative, are equal (this is the cocycle condition), then there exists an object t over T and morphisms $t_\alpha \to t$ such that the composites $t_{\alpha\beta} \to t_\alpha \to t$ and $t_{\alpha\beta} \to t_\beta \to t$ are equal, for every α and β.

3. Stacks from groupoid schemes

Given a groupoid scheme $R \rightrightarrows U$, we saw that the associated CFG $[R \rightrightarrows U]^{\mathrm{pre}}$ is a prestack. For instance, Example 3.12 shows that $[G \rightrightarrows \Lambda]^{\mathrm{pre}}$ is equivalent to the category of trivial G-torsors. However $[R \rightrightarrows U]^{\mathrm{pre}}$ is not a stack in general. A general G-torsor is only locally trivial, which means that Axiom 2 will most certainly fail.

Since G-torsors are a topic that is familiar to us, we know how to proceed in this case. We enlarge the category to contain all the *locally trivial* G-torsors, and we obtain the category BG which is a stack (at least, e.g., when G is an affine group scheme over the base). In this section, we imitate this construction for a general groupoid scheme. There will be a notion of $(R \rightrightarrows U)$-torsor. This will be a stack when the groupoid satisfies a hypothesis that allows a result from the theory of descent to be applied, akin to requiring G to be affine (or at least quasi-affine) over the base scheme. We emphasize that notion of $(R \rightrightarrows U)$-torsor is supposed to be parallel to (and a generalization of) the notion of G-torsor.

Consider the prestack $[R \rightrightarrows U]^{\mathrm{pre}}$. A descent datum, in the setting of Axiom $2'$, consists of a morphism $\varphi\colon T' \to U$ (object over T') and a morphism $\Phi\colon T'' \to R$ satisfying $s \circ \Phi = \varphi \circ p_1$ and $t \circ \Phi = \varphi \circ p_2$ (morphism from the pullback by $p_1\colon T'' \to T'$ to the pullback by $p_2\colon T'' \to T'$). This must satisfy the cocycle condition, and it can be checked that this precisely says that (φ, Φ) is a morphism of groupoid schemes from $T'' \to T'$ to $R \rightrightarrows U$. One idea for producing a stack would be to enlarge the category by admitting, as objects, any covering $T' \to T$ together with morphism of groupoid schemes from $T'' \to T'$ to $R \rightrightarrows U$. Then one faces the challenging task of describing morphisms between objects, and compositions of morphisms (see Remark 4.25, below). We choose a different approach, one which constrains the choice of $T' \to T$. For instance, in the case of the groupoid scheme $G \rightrightarrows \Lambda$, the T' will have to be a G-torsor E with projection to T. This means, then, that T'' can be canonically identified with $E \times G$ (making the projections maps first projection pr_1 and action a). The point is that the resulting morphism of groupoid schemes

$$\begin{tikzcd} E \times G \arrow[r] \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "a"] & G \arrow[d, shift right] \arrow[d, shift left] \\ E \arrow[r] & \Lambda \end{tikzcd}$$

yields a cartesian diagram when either the left-hand vertical maps or the right-hand vertical maps are selected. In rough terms, the category of $(R \rightrightarrows U)$-torsors will be the category whose objects are pairs consisting of a morphism of schemes $T' \to T$ and a morphism of groupoid schemes from $T'' := T' \times_T T' \rightrightarrows T'$ to $R \rightrightarrows U$, where the latter is required to give rise to a cartesian diagram by selecting either the pair of source maps or the pair of target maps. As in the case of G-torsors, a morphism over $S \to T$

will be a morphism $S' \to T'$ satisfying some compatibility properties (akin to the G-equivariance requirement). Also as in the case of G-torsors, there will be an additional local triviality requirement.

Definition 4.5. Consider a morphism (ϕ, Φ) of groupoid schemes from $R' \rightrightarrows U'$ to $R \rightrightarrows U$:

$$\begin{tikzcd} R' \arrow[r, "\Phi"] \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ U' \arrow[r, "\phi"] & U \end{tikzcd}$$

We say that (ϕ, Φ) is a **square morphism** of groupoid schemes if the diagrams

$$\begin{tikzcd} R' \arrow[r, "\Phi"] \arrow[d, "s'"'] & R \arrow[d, "s"] \\ U' \arrow[r, "\phi"] & U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R' \arrow[r, "\Phi"] \arrow[d, "t'"'] & R \arrow[d, "t"] \\ U' \arrow[r, "\phi"] & U \end{tikzcd}$$

are cartesian.

Notice that since either a source map or a target map can be obtained from the other by composing with the inverse map, if one of these diagram is cartesian, then the other must be cartesian as well.

Definition 4.6. Let $g\colon T' \to T$ be a morphism of schemes. A **banal groupoid scheme** for g is a groupoid scheme $p, q\colon T'' \rightrightarrows T'$ such that $g \circ p = g \circ q$ and such that $(p, q)\colon T'' \to T' \times_T T'$ is an isomorphism.

In other words, $T'' \rightrightarrows T'$ is one of the groupoid schemes that we saw arising from a morphism of schemes in Example 3.1. Notice that, given morphisms p and q from T'' to T' such that $g \circ p = g \circ q$ and $(p, q)\colon T'' \xrightarrow{\sim} T' \times_T T'$, there is then a unique groupoid scheme structure on $T'' \rightrightarrows T'$ having structure morphisms p and q.

Example 4.7. Let $R \rightrightarrows U$ be a groupoid scheme. There is then a square morphism of groupoid schemes

$$\begin{tikzcd} R \,{}_t\!\times_s R \arrow[r, "\mathrm{pr}_2"] \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "m"] & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ R \arrow[r, "t"] & U \end{tikzcd}$$

The groupoid scheme on the left is a banal groupoid scheme for the morphism $s\colon R \to U$.

Exercise 4.2. Verify the assertions of Example 4.7. [Hint: use the fact mentioned in Exercise 3.1(d), that the diagrams of Axiom (2) for groupoid scheme are cartesian.]

Exercise 4.3. Let $s, t\colon R \rightrightarrows U$ be a groupoid scheme, X a scheme, and $f\colon U \to X$ a morphism such that $f \circ s = f \circ t$. If $g\colon X' \to X$ is any morphism, with banal groupoid scheme $p, q\colon X'' \rightrightarrows X'$ for g, then any lift $\phi\colon U \to X'$ of f extends uniquely to a morphism of groupoid schemes (ϕ, Φ) from $R \rightrightarrows U$ to $X'' \rightrightarrows X'$.

We describe a category of $(R \rightrightarrows U)$-torsors. Then $[R \rightrightarrows U]$ will be defined as the full subcategory of objects satisfying a local triviality requirement.

Preliminary definition. A $(R \rightrightarrows U)$-torsor will consist of a morphism $g\colon T' \to T$, a banal groupoid $T'' \rightrightarrows T'$ for g, and a square morphism of groupoid schemes (γ, Γ) from $T'' \rightrightarrows T'$ to $R \rightrightarrows U$. These form a category, fibered in groupoids over the base category of schemes. If we have another object, $h\colon S' \to S$ with $S'' \rightrightarrows S'$ and morphism (ξ, Ξ) to $R \rightrightarrows U$, then a morphism between these objects over $f\colon S \to T$ will consist of a morphism $\phi\colon S' \to T'$ satisfying two conditions.

(i) The morphism ϕ fits into a cartesian diagram

$$\begin{tikzcd} S' \arrow[r, "\phi"] \arrow[d, "h"'] & T' \arrow[d, "g"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

Hence, by Exercise 4.3, there is then a unique $\Phi\colon S'' \to T''$ making (ϕ, Φ) a morphism of groupoid schemes.

(ii) We have $\gamma \circ \phi = \xi$ and $\Gamma \circ \Phi = \Xi$, in other words, a commutative diagram of groupoid schemes

$$\begin{tikzcd} S'' \arrow[rr, "\Xi"] \arrow[dr, "\Phi"'] \arrow[dd, shift left] \arrow[dd, shift right] & & R \arrow[dd, shift right, "s"'] \arrow[dd, shift left, "t"] \\ & T'' \arrow[ur, "\Gamma"'] \arrow[dd, shift left] \arrow[dd, shift right] & \\ S' \arrow[rr, "\xi"] \arrow[dr, "\phi"'] & & U \\ & T' \arrow[ur, "\gamma"'] & \end{tikzcd}$$

As in earlier examples, the verification of the CFG axioms for this category is based on the existence and universal property of the fiber product of schemes. (Given an object $g\colon T' \to T$, etc., and an arbitrary morphism $f\colon S \to T$, set $S' = S \times_T T'$ with projection maps h to S and ϕ to T', and $S'' = S \times_T T''$, and take ξ to be $\gamma \circ \phi$.) We point out, that in this preliminary definition there is no mention yet of the topology on the base category. Our next task is to impose a local triviality hypothesis on objects, using the Grothendieck topology of $\mathcal{S}$.

Example 4.8. The scheme U, map $s\colon R \to U$, and square morphism of groupoid schemes of Example 4.7 constitute an $(R \rightrightarrows U)$-torsor (over the scheme U).

Definition 4.9. Given a pair consisting of a banal groupoid scheme $g\colon T' \to T$ and a square morphism of groupoid schemes (γ, Γ) from $T'' \to T'$ to $R \rightrightarrows U$, we say that this object is

(i) **trivial** if there exists a morphism to the object of Example 4.8 (over some morphism $T \to U$), and

(ii) **locally trivial** if there exists a trivial object $h\colon S' \to S$ with $S'' \rightrightarrows S'$ and morphism (ξ, Ξ) to $R \rightrightarrows U$, a covering map $f\colon S \to T$, and a morphism from the trivial object to the given object over this morphism f.

Definition 4.10. The CFG $[R \rightrightarrows U]$ is defined to be the category of locally trivial $(R \rightrightarrows U)$-torsors. Specifically, it is the full subcategory of the category in the Preliminary definition, consisting of objects that are locally trivial (Definition 4.9).

For the sake of brevity, we will speak of $(R \rightrightarrows U)$-torsors with the understanding that they will always be locally trivial, i.e., objects of $[R \rightrightarrows U]$. The notion of locally trivial makes sense over an arbitrary site (base category equipped with a Grothendieck topology). Of course, for us this will usually be a category of schemes with the étale topology.

In the case $R \rightrightarrows U$ is $G \rightrightarrows \Lambda$, we see that a $(G \rightrightarrows \Lambda)$-torsor is just a G-torsor $E \to S$, together with a choice of product $E \times G$ (which, by the projection map and action map, is isomorphic to $E \times_S E$). This choice is not significant: the same G-torsor with a different choice of product $\widetilde{E \times G}$ is canonically isomorphic (in the CFG $[R \rightrightarrows U]$) to $E \times G$ by the morphism $\phi = 1_E$. The $(G \rightrightarrows \Lambda)$-torsor $E \to S$ is trivial precisely when E is G-equivariantly isomorphic to $S \times G$, and the locally triviality condition in Definition 4.9(ii) is the same as the usual local triviality condition on a G-torsor (cf. Example 2.3).

Just as in the case of G-torsors, we will be able to prove that $[R \rightrightarrows U]$ is a stack, provided that the groupoid scheme $R \rightrightarrows U$ satisfies some hypothesis that will enable us to invoke an appropriate result from the theory of descent. We want to use descent for affine schemes (Proposition A.12), or more generally quasi-affine schemes (Proposition A.17). It turns out to be most natural to place the hypothesis on the *relative diagonal* of the groupoid $(s,t)\colon R \to U \times U$. The reason is that given another groupoid scheme $R' \rightrightarrows U'$ with a morphism to $R \rightrightarrows U$ that is supposed to be an equivalence of groupoids, the relative diagonal $R' \to U' \times U'$ will be a pullback of $R \to U \times U$ (see Condition 1.3(i)). To put this in a concrete setting, the CFG $\underline{X}$ (where X is a scheme) will be represented by the groupoid $U \times_X U \rightrightarrows U$ for any covering $U \to X$. The properties (separated, affine, etc.) of an individual projection map $U \times_X U \to U$ depend heavily on the choice of U. Whereas, the relative diagonal $U \times_X U \to U \times U$ is gotten by base change from the absolute diagonal $X \to X \times X$, and hence inherits any property that is preserved by base change (e.g., it is always separated and is proper precisely when X is separated, regardless of the choice of U). For this reason, we focus on properties of the relative diagonal.

Proposition 4.11. *Let $R \rightrightarrows U$ be a groupoid scheme. Assume that the relative diagonal $(s,t)\colon R \to U \times U$ is quasi-affine. Then $[R \rightrightarrows U]$ is a stack (for the étale topology on the base category of schemes).*

In this situation we will call $[R \rightrightarrows U]$ the *stack of* $(R \rightrightarrows U)$*-torsors*. To maintain the analogy with ordinary torsors, we will denote a typical object of $[R \rightrightarrows U]$ by $E \to T$ (of course an object will be understood to include a choice of product $E \times_T E$ and a square morphism of groupoids from $E \times_T E \rightrightarrows E$ to $R \rightrightarrows U$).

Proof. Let $E \to T$ be an object of $[R \rightrightarrows U]$. Then, we claim, E is quasi-affine over $T \times U$. To address this claim, let us denote by g the morphism $E \to T$ and by (γ, Γ) the morphism from $E \times_T E \rightrightarrows E$ to $R \rightrightarrows U$. The morphism $(g,\gamma)\colon E \to T \times U$, it is claimed, is quasi-affine. Let $f\colon S \to T$ be a covering map, $h\colon D \to S$ an object of $[R \rightrightarrows U]$ over S, and suppose a morphism φ in $[R \rightrightarrows U]$ from this object to the object $E \to T$ is given. Then there is a diagram

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[d, "h"'] & E \arrow[r, "\gamma"] \arrow[d, "g"] & U \\ S \arrow[r, "f"'] & T & \end{tikzcd}$$

in which the square is cartesian. As a consequence, we have a cartesian diagram

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[d, "(h,\gamma\circ\varphi)"'] & E \arrow[d, "(g,\gamma)"] \\ S \times U \arrow[r, "f\times 1_U"] & T \times U \end{tikzcd}$$

The bottom map is a covering map. Hence (g, γ) is quasi-affine if and only if $(h, \gamma \circ \varphi)$ is quasi-affine. So, to establish this claim, we are reduced to the case of a trivial object. By the definition of triviality, there is a cartesian diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "(g,\gamma)"'] & R \arrow[d, "(s,t)"] \\ T \times U \arrow[r] & U \times U \end{tikzcd}$$

By hypothesis, (s, t) is quasi-affine, hence so is (g, γ), and the claim is established.

Now the proof of the proposition uses effective descent for schemes quasi-affine over a given scheme (Proposition A.17). Consider an étale cover $T' \to T$, an object $E' \to T'$ of $[R \rightrightarrows U]$, and gluing data on $T'' := T' \times_T T'$ that satisfies the cocycle condition. Then $E' \to T' \times U$ is quasi-affine, and we have a descent datum which, by effective descent, produces a scheme E quasi-affine over $T \times U$, together with a compatible isomorphism $T' \times_T E \cong E$. We focus on $E \to T$. We have also $E \to U$, and by invoking descent for morphisms to a given target (Proposition A.13), we recover a morphism $E \times_T E \to R$ whose composite with $E' \times_{T'} E' \to E \times_T E$ is the given morphism $E' \times_{T'} E' \to R$. Now it must be checked that the morphism from $E \times_T E \rightrightarrows E$ to $R \rightrightarrows U$ is a square morphism of groupoid schemes. To be a morphism of groupoid schemes involves checking some identities of morphisms. Again we know this after replacing each source by a covering, so we have the desired identities by invoking now the uniqueness portion of Proposition A.13. To be a square morphism of groupoid schemes, now, is that a certain morphism to a fiber product is an isomorphism, when we know this holds after étale base change. We need a result that says a morphism, which after étale base change becomes an isomorphism, must itself be an isomorphism. This is so by [**EGA** IV.2.7.1] (which tells us that this is true for base change by a faithfully flat quasi-compact morphism, and using that an étale morphism is open we easily deduce that it is true for base change by an étale morphism). □

The hypothesis (s,t) quasi-affine is satisfied in many common situations. We list a few of these. First, whenever s and t are étale, and (s,t) is quasi-compact and separated, then it is quasi-affine; this is because any quasi-finite separated morphism is quasi-affine [**EGA** IV.18.12.12]. If s and t are themselves quasi-affine, then (s,t) will be quasi-affine provided R is quasi-separated, because it can be factored through $R \times R$. For instance, the groupoid scheme $X \times G \rightrightarrows X$ arising from a group action has this property whenever G is a group scheme, quasi-affine over Λ, and X is quasi-separated over Λ. (Any locally Noetherian scheme is quasi-separated over an arbitrary base scheme, so this applies to most group actions met in practice.)

4. Stackification via torsors

Let $R \rightrightarrows U$ be a groupoid scheme. There is a natural morphism of CFGs (defined up to canonical 2-isomorphism) from $[R \rightrightarrows U]^{\mathrm{pre}}$ to $[R \rightrightarrows U]$. To an object $g\colon T \to U$ in $[R \rightrightarrows U]^{\mathrm{pre}}$ there is an associated trivial $(R \rightrightarrows U)$-torsor:

(1)
$$\begin{tikzcd} T \,{}_g{\times}_s\, R \,{}_t{\times}_s\, R \arrow[r, "\mathrm{pr}_3"] \arrow[d, shift right, "\mathrm{pr}_{12}"'] \arrow[d, shift left, "1_T \times m"] & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ T \times_{U,s} R \arrow[r, "t \circ \mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & U \\ T & \end{tikzcd}$$

To a morphism $\gamma\colon T' \to R$ in $[R \rightrightarrows U]^{\mathrm{pre}}$, over $f\colon T' \to T$, between objects $g'\colon T' \to U$ and $g\colon T \to U$ (we recall, this means $s \circ \gamma = g'$ and $t \circ \gamma = g \circ f$), there is the following morphism in $[R \rightrightarrows U]$:

(2)
$$(f \circ \mathrm{pr}_1, m((i \circ \gamma) \times 1_R))\colon T' \times_{U,s} R \to T \times_{U,s} R.$$

In the case $[G \rightrightarrows \Lambda]$, this associates to an object T (i.e., $T \to \Lambda$) the trivial G-torsor $T \times G \to T$. A morphism $\gamma\colon T \to G$ over 1_T is sent to the G-equivariant isomorphism $T \times G \to T \times G$, $(x,g) \mapsto (x, \gamma(x)^{-1}g)$. (Compare with Example 3.9.)

Definition 4.12. Let $\mathfrak{X}_0$ be a CFG$/\mathcal{S}$. Then a **stackification** of $\mathfrak{X}_0$ is a stack $\mathfrak{X}$ together with a morphism of CFGs $b\colon \mathfrak{X}_0 \to \mathfrak{X}$ such that for any stack $\mathfrak{Y}$ the functor

$$\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y}) \to \mathrm{HOM}(\mathfrak{X}_0, \mathfrak{Y})$$

induced by composition with b is an equivalence of categories.

We wish to assert that the stack $[R \rightrightarrows U]$ is a stackification of $[R \rightrightarrows U]^{\mathrm{pre}}$. This means, in particular, that any given morphism $f_0\colon [R \rightrightarrows U]^{\mathrm{pre}} \to \mathfrak{Y}$ should, up to 2-isomorphism, be the composition of a morphism $f\colon [R \rightrightarrows U] \to \mathfrak{Y}$ with the morphism $b\colon [R \rightrightarrows U]^{\mathrm{pre}} \to [R \rightrightarrows U]$ defined in (1) and (2). We can give an informal description of such a morphism f. Let $E \to T$ be an $(R \rightrightarrows U)$-torsor. By local triviality, there exists a covering map $T' \to T$ and a an isomorphism between a trivial torsor on T' and the pullback $T' \times_T E$. The trivial torsor comes from a morphism $T' \to U$, i.e., is $b(t')$ for some object t' of $[R \rightrightarrows U]^{\mathrm{pre}}$ over T'. Set $y' = f_0(t')$. On $T'' := T' \times_T T'$ we have isomorphisms of the pullback of $b(t')$, by either projection map, with $T'' \times_T E$. Composing these, we get an isomorphism $b(p_1^* t') \to b(p_2^* t')$. A crucial fact below is that b is, as a functor, *fully faithful*. So this isomorphism comes from an isomorphism $\varphi\colon p_1^* t' \to p_2^* t'$, which must satisfy the cocycle condition since the cocycle condition holds after applying b. Hence y' and $f_0(\varphi)$ constitute a descent datum in $\mathcal{Y}$. Since $\mathcal{Y}$ is a stack, this descent datum is effective, meaning y' is identified with a pullback of some object y in $\mathcal{Y}_T$. There is a functor f, sending $E \to T$ to y (it is of course also necessary to specify how f acts on morphisms), and this is the desired morphism of stacks.

Turning this rough description into an assertion with proof is the next task. We start with a preliminary result. It will be convenient to use the restatement of the stack axioms of Remark 4.4.

Proposition 4.13. *Let $\mathfrak{X}_0$ be a CFG, let $\mathfrak{X}$ be a stack, and let $b\colon \mathfrak{X}_0 \to \mathfrak{X}$ be a morphism of CFGs which, as a functor, is full and faithful. If, for every object T in $\mathcal{S}$ and every object x of $\mathfrak{X}$ over T there exists a covering family $\{T_\alpha \to T\}$ and for every α an object t_α in $\mathfrak{X}_0$ and a morphism $b(t_\alpha) \to x$ over $T_\alpha \to T$, then $\mathfrak{X}$ is a stackification of $\mathfrak{X}_0$ (by the morphism b).*

Proof. To show that $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y}) \to \mathrm{HOM}(\mathfrak{X}_0, \mathfrak{Y})$ is essentially surjective, we suppose that $f_0\colon \mathfrak{X}_0 \to \mathfrak{Y}$ is given. Let x be as in the statement of the proposition. So there is a covering family $\{T_\alpha \to T\}$, with morphisms $b(t_\alpha) \to x$. Set $T_{\alpha\beta} = T_\alpha \times_T T_\beta$ and $T_{\alpha\beta\gamma} = T_\alpha \times_T T_\beta \times_T T_\gamma$. Choose $t_{\alpha\beta} \to t_\alpha$ over $T_{\alpha\beta} \to T_\alpha$. By the CFG axioms there is a unique morphism $b(t_{\alpha\beta}) \to b(t_\beta)$ over $T_{\alpha\beta} \to T_\beta$ whose composite with the morphism to x is equal to the composite $b(t_{\alpha\beta}) \to b(t_\alpha) \to x$, and since b is fully faithful this is the image under b of a morphism $t_{\alpha\beta} \to t_\beta$. Now let $t_{\alpha\beta\gamma} \to t_{\alpha\beta}$ be any morphism over $T_{\alpha\beta\gamma} \to T_{\alpha\beta}$ and let $t_{\alpha\beta\gamma} \to t_{\beta\gamma}$ be the unique morphism over $T_{\alpha\beta\gamma} \to T_{\beta\gamma}$ making a commutative diagram with t_β. The unique morphism $t_{\alpha\beta\gamma} \to t_{\alpha\gamma}$ over $T_{\alpha\beta\gamma} \to T_{\alpha\gamma}$ making a commutative diagram with t_α transforms via b to the unique morphism $b(t_{\alpha\beta\gamma}) \to b(t_{\alpha\gamma})$ whose composite with the morphism to x is equal to $b(t_{\alpha\beta\gamma}) \to b(t_{\alpha\beta}) \to x$. This in turn is equal to the composite $b(t_{\alpha\beta\gamma}) \to b(t_{\beta\gamma}) \to x$, equal to the image under b of the unique morphism $t_{\alpha\beta\gamma} \to t_{\alpha\gamma}$ making a commutative diagram with t_γ. So we have a commutative diagram

$$\begin{tikzcd}
 & t_{\alpha\beta\gamma} \arrow[dl] \arrow[d] \arrow[dr] & \\
t_{\alpha\beta} \arrow[d] \arrow[dr] & t_{\alpha\gamma} \arrow[dl] \arrow[dr] & t_{\beta\gamma} \arrow[dl] \arrow[d] \\
t_\alpha & t_\beta & t_\gamma
\end{tikzcd}$$

If we apply f_0 we get a similar commutative diagram in $\mathfrak{Y}$. Since $\mathfrak{Y}$ is a stack, it follows from Axiom 2 (restated, as in Remark 4.4) that there is an object y, together with morphisms $f_0(t_\alpha) \to y$, making commutative diagrams with the $f_0(t_{\alpha\beta})$. We define $f(x) = y$.

It remains to specify how morphisms transform under f, and to supply a natural isomorphism $f \circ b \Rightarrow f_0$. Let $S \to T$ be a morphism, x an object of $\mathfrak{X}$ over T, and u

an object of $\mathfrak{X}$ over S. Suppose $f(x) = y$, with $\{T_\alpha \to T\}$, t_α, $t_{\alpha\beta} \to t_\alpha$, $t_{\alpha\beta} \to t_\beta$, and $f_0(t_\alpha) \to y$ are as above. Suppose $f(u) = v$, with $\{S_\gamma \to S\}$, s_γ, $s_{\gamma\delta} \to s_\gamma$, $s_{\gamma\delta} \to s_\delta$, and $f_0(s_\gamma) \to v$, analogously. Now consider a morphism $u \to x$ over $S \to T$. We define $f(u \to x)$ to be the morphism $v \to y$ characterized as follows. For each α and γ let $S_{\alpha\gamma\varepsilon}$ be a collection of objects over $S_\gamma \times_T T_\alpha$ (with ε in an indexing set that depends on α and γ) such that for each fixed γ the collection $\{S_{\alpha\gamma\varepsilon} \to S_\gamma\}$ is a covering family. An example of such a family is $\{S_\gamma \times_T T_\alpha \to S_\gamma\}$. For each α, γ, and ε, let $s_{\alpha\gamma\varepsilon} \to s_\gamma$ be a morphism over $S_{\alpha\gamma\varepsilon} \to S_\gamma$ Now there is a unique morphism $b(s_{\alpha\gamma\varepsilon}) \to b(t_\alpha)$ over $S_{\alpha\gamma\varepsilon} \to T_\alpha$ whose composite with $b(t_\alpha) \to x$ is equal to the composite $b(s_{\alpha\gamma\varepsilon}) \to b(s_\gamma) \to u \to x$. As b is fully faithful this is b applied to a unique morphism $s_{\alpha\gamma\varepsilon} \to t_\alpha$. We may apply f_0 and compose with the morphism $f_0(t_\alpha) \to y$ to get a morphism

$$f_0(s_{\alpha\gamma\varepsilon}) \to y.$$

This, we claim, gives rise to a unique morphism $v \to y$ by Axiom 1, restated as in Remark 4.4; this we take to be $f(u \to x)$.

To verify this we consider $S_{\alpha\gamma\varepsilon\beta\delta\eta} := S_{\alpha\gamma\varepsilon} \times_S S_{\beta\delta\eta}$, with $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to s_{\alpha\gamma\varepsilon}$ lying over the first projection map and as usual a unique morphism $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to s_{\beta\delta\eta}$ whose image under b makes a commutative diagram with v. This last morphism can be obtained by starting with the unique morphism $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to s_{\gamma\delta}$ over $S_{\alpha\gamma\varepsilon\beta\delta\eta} \to S_{\gamma\delta}$ making a commutative diagram with s_γ, and then selecting the unique morphism over $S_{\alpha\gamma\varepsilon\beta\delta\eta} \to S_{\alpha\gamma\varepsilon}$ making the diagram below with s_δ commute

$$\begin{tikzcd} s_{\gamma\delta} \arrow[d] & s_{\alpha\gamma\varepsilon\beta\delta\eta} \arrow[l, dashed] \arrow[d] \\ s_\gamma & s_{\alpha\gamma\varepsilon} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} s_{\gamma\delta} \arrow[d] & s_{\alpha\gamma\varepsilon\beta\delta\eta} \arrow[l] \arrow[d, dashed] \\ s_\delta & s_{\beta\delta\eta} \arrow[l] \end{tikzcd}$$

Then a diagram chase shows that after applying f_0 it gives a commutative diagram with v. To be able to apply Axiom 1 to get a morphism $v \to y$, we must verify that the two morphisms $f_0(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \to y$ in the following diagram, going via $f_0(s_{\alpha\gamma\varepsilon})$ and via $f_0(s_{\beta\delta\eta})$, are equal:

$$\begin{tikzcd} & f_0(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \arrow[dl] \arrow[d] \arrow[dr, dashed] & \\ f_0(s_{\alpha\gamma\varepsilon}) \arrow[dr] & f_0(s_{\beta\delta\eta}) \arrow[dr] & f_0(t_{\alpha\beta}) \arrow[dl] \arrow[d] \\ & f_0(t_\alpha) \arrow[d] & f_0(t_\beta) \arrow[dl] \\ & y & \end{tikzcd}$$

There is a unique morphism $b(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \to b(t_{\alpha\beta})$ whose composite with the morphism to x is equal to the composite $b(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \to b(s_{\gamma\delta}) \to u \to x$. This morphism is b applied to some morphism $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to t_{\alpha\beta}$. If we can show that the composite with the map to t_α is equal to $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to s_{\alpha\gamma\varepsilon} \to t_\alpha$, and analogously for the maps to t_β, then the dotted arrow above will make both of the upper squares in the diagram commute and the required equality of morphisms will follow. By faithfulness it suffices to verify these assertions after applying b. For the first assertion, both composites, further composed with $b(t_\alpha) \to x$, are equal to the composite $b(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \to b(s_\gamma) \to u \to x$, and hence they are equal. Similarly, both composites to $b(t_\beta)$ are equal.

We claim that the morphism $v \to y$ thus produced is independent of the choice of covering families $\{S_{\alpha\gamma\varepsilon} \to S_\gamma\}$. For this, it suffices to check that the morphism produced is unchanged by refinement (i.e., replacing $S_{\alpha\gamma\varepsilon} \to S_\gamma$ by $S_{\alpha\gamma\varepsilon\lambda} \to S_{\alpha\gamma\varepsilon} \to S_\gamma$, where $\{S_{\alpha\gamma\varepsilon\lambda} \to S_{\alpha\gamma\varepsilon}\}$ is a covering family) and also unchanged by a change of maps to the T_α, i.e., re-indexing $S_{\alpha\gamma\varepsilon}$ as $S_{\alpha\beta\gamma\lambda}$ and replacing $S_{\alpha\beta\gamma\lambda} \to T_\alpha$ with some other map $S_{\alpha\beta\gamma\lambda} \to T_\beta$. We leave the case of refinements as an exercise: the first step is that, if we choose $s_{\alpha\gamma\varepsilon\lambda} \to s_{\alpha\gamma\varepsilon}$ over $S_{\alpha\gamma\varepsilon\lambda} \to S_{\alpha\gamma\varepsilon}$, then the morphism $b(s_{\alpha\gamma\varepsilon\lambda}) \to b(t_\alpha)$ that is stipulated above is the composite $b(s_{\alpha\gamma\varepsilon\lambda}) \to b(s_{\alpha\gamma\varepsilon}) \to b(t_\alpha)$. Let us treat changes of maps to the T_α. The maps $S_{\alpha\beta\gamma\lambda} \to T_\alpha$ and $S_{\alpha\beta\gamma\lambda} \to T_\beta$ determine a map $S_{\alpha\beta\gamma\lambda} \to T_{\alpha\beta}$. There is a unique morphism $b(s_{\alpha\beta\gamma\lambda}) \to b(t_{\alpha\beta})$ whose composite with the map to x is equal to the composite $b(s_{\alpha\beta\gamma\lambda}) \to b(s_\gamma) \to u \to x$. Therefore the composite morphism $b(s_{\alpha\beta\gamma\lambda}) \to b(t_{\alpha\beta}) \to b(t_\alpha)$ is the morphism stipulated in the construction. Analogously, the composite morphism to $b(t_\beta)$ is as stipulated in the construction applied to $S_{\alpha\beta\gamma\lambda} \to T_\beta$. The maps $f_0(s_{\alpha\beta\gamma\lambda}) \to f_0(t_\alpha) \to y$ and $f_0(s_{\alpha\beta\gamma\lambda}) \to f_0(t_\beta) \to y$ are then equal, since they both factor through $f_0(t_{\alpha\beta})$.

We have, indeed, produced a functor f. Indeed, if $w \to u$ is a morphism in $\mathfrak{X}_0$ over $R \to S$, with image $z \to v$ in $\mathfrak{Y}$, then the composite $z \to v \to y$ is seen to satisfy the criteria characterizing the image under f of the composite $w \to u \to x$. For, if $\{R_\pi \to R\}$ is a covering family, with $R_{\gamma\pi\iota} \to R_\pi \times_S S_\gamma$, then we take $R_{\alpha\gamma\varepsilon\pi\iota}$ over $R_{\gamma\pi\iota} \times_{S_\gamma} S_{\gamma\alpha\varepsilon}$ such that for every fixed π, γ, and ι, $\{R_{\alpha\gamma\varepsilon\pi\iota} \to R_{\gamma\pi\iota}\}$ is a covering family. We use the characterization of $f(w \to x)$ coming from the $R_{\alpha\gamma\varepsilon\pi\iota}$. Let us suppose $f(w \to u)$ is constructed using $r_{\gamma\pi\iota} \to r_\pi$ and $r_{\gamma\pi\iota} \to s_\gamma$. Now choose $r_{\alpha\gamma\varepsilon\pi\iota} \to r_{\gamma\pi\iota}$ over $R_{\alpha\gamma\varepsilon\pi\iota} \to R_{\gamma\pi\iota}$, and let $r_{\alpha\gamma\varepsilon\pi\iota} \to s_{\alpha\gamma\varepsilon}$ be the unique morphism over $R_{\alpha\gamma\varepsilon\pi\iota} \to S_{\alpha\gamma\varepsilon}$ whose composite with $s_{\alpha\gamma\varepsilon} \to s_\gamma$ is equal to the composite $r_{\alpha\gamma\varepsilon\pi\iota} \to r_{\gamma\pi\iota} \to s_\gamma$. We have

$$\begin{tikzcd}
 & & r_{\alpha\gamma\varepsilon\pi\iota} \arrow[dl] \arrow[dr] & & \\
 & r_{\gamma\pi\iota} \arrow[dl] \arrow[dr] & & s_{\alpha\gamma\varepsilon} \arrow[dl] \arrow[dr] & \\
r_\pi & & s_\gamma & & t_\alpha
\end{tikzcd}$$

which, upon applying b, commutes with $w \to u \to x$, and upon applying f_0, commutes with $z \to v \to y$, so the latter is equal to $f(w \to x)$. It follows immediately from the construction that $f(1_x) = 1_y$. When $x = b(t)$ we have $f_0(b(t))$ in $\mathfrak{Y}$, with maps $f_0(t_\alpha) \to f_0(b(t))$ such that both composites $f_0(t_{\alpha\beta}) \to f_0(b(t))$ are equal. Hence there is a unique isomorphism $f_0(b(t)) \to y$ compatible with the morphisms from the $f_0(t_\alpha)$. If $u = b(s)$ and we have a morphism $u \to x$ equal to b applied to some $s \to t$, then the composite $v \xrightarrow{\sim} f_0(b(s)) \to f_0(b(t)) \xrightarrow{\sim} y$ satisfies the criterion which characterizes $f(u \to x)$. Hence we have a natural isomorphism $f_0 \circ b \Rightarrow f$.

We now show that the functor between HOM categories is fully faithful. Let f and g be morphisms $\mathfrak{X} \to \mathfrak{Y}$, and let $f_0 = f \circ b$ and $g_0 = g \circ b$. Given a natural isomorphism $f_0 \Rightarrow g_0$ we need to show it is produced by a unique natural isomorphism $f \Rightarrow g$. Let $b(t_\alpha) \to x$ be as in the hypothesis. If we set $y = f(x)$ and $z = g(x)$

then the given natural isomorphism yields morphisms $f_0(t_\alpha) \to g_0(t_\alpha) \to z$. With $t_{\alpha\beta} \to t_\alpha$ and $t_{\alpha\beta} \to t_\beta$ such that the composite morphisms to x are equal, we have the composite $f_0(t_{\alpha\beta}) \to f_0(t_\alpha) \to z$ equal to $f_0(t_{\alpha\beta}) \to g_0(t_{\alpha\beta}) \to z$, which is equal to $f_0(t_{\alpha\beta}) \to f_0(t_\beta) \to z$. So, by Axiom 1, restated as in Remark 4.4, there is a uniquely determined isomorphism $y \to z$. Naturality is the condition that $v \to y \to z$ is equal to $v \to w \to z$, where $u \to x$ is a morphism over some $S \to T$, and $v = f(u)$ and $w = g(u)$. We introduce $S_{\alpha\gamma\varepsilon}$ and $s_{\alpha\gamma\varepsilon}$ as above; notice that the image $v \to y$ of $u \to x$ under f has the property that the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to y$ is equal to the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(t_\alpha) \to y$, and that a similar assertion holds with g_0 in place of f_0 and $w \to z$ in place of $v \to y$. To verify naturality, it suffices by Axiom 1 to verify that the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to y \to z$ is equal to the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to w \to z$, for every α, γ, and ε This is now a routine diagram chase, using the naturality of $f_0 \Rightarrow g_0$ and the fact that the morphism $y \to z$, resp. $v \to w$, is characterized by its fitting in a commutative diagram with $f_0(t_\alpha) \to g_0(t_\alpha)$, resp. with $f_0(s_\gamma) \to g_0(s_\gamma)$. □

Exercise 4.4. Supply the details to the argument that the morphism $v \to y$ (image of $u \to x$ under f) is unchanged by refinement.

Proposition 4.14. *Let $R \rightrightarrows U$ be a groupoid scheme with quasi-affine relative diagonal $(s,t)\colon R \to U \times U$. Then the morphism $b\colon [R \rightrightarrows U]^{\mathrm{pre}} \to [R \rightrightarrows U]$ defined in* (1) *and* (2) *is a stackification.*

Proof. We use Proposition 4.13. By the definition of $[R \rightrightarrows U]$, every object is locally isomorphic to an object in the image of b. So, it remains only to show that b is fully faithful. Morphisms in $[R \rightrightarrows U]^{\mathrm{pre}}$ from $g'\colon T' \to U$ to $g\colon T \to U$ over $f\colon T' \to T$ are morphisms $\gamma\colon T' \to R$ satisfying $s \circ \gamma = g'$ and $t \circ \gamma = g \circ f$. In $[R \rightrightarrows U]$, the morphisms over f from $b(g')$ to $b(g)$ are morphisms of the form

$$(f \circ \mathrm{pr}_1, \delta)\colon T' \times_U R \to T \times_U R$$

where δ is required to satisfy $s \circ \delta = g \circ f \circ \mathrm{pr}_1$ (obvious condition to have target $T \times_U R$), $t \circ \delta = t \circ \mathrm{pr}_2$ (condition (i) to be a morphism), and $m(\delta \times 1_R) = \delta \circ (1_{T'} \times m)$ (condition (ii) to be a morphism). The functor b sends γ to $(f \circ \mathrm{pr}_1, m((i \circ \gamma) \times 1_R))$. We define a map the other way sending $(f \circ \mathrm{pr}_1, \delta)$ to $i \circ \delta \circ (1_{T'}, e \circ g')$. Composing the two maps in one order we have γ mapping to $i \circ m(i \circ \gamma, e \circ g')$, which is equal to γ. In the other order, $(f \circ \mathrm{pr}_1, \delta)$ is sent to the map $T' \times_U R \to T \times_U R$ which, on the first factor is $f \circ \mathrm{pr}_1$ and on the second factor is $m((\delta \circ (1_{T'}, e \circ g')) \times 1_R) = \delta \circ (\mathrm{pr}_1, m((e \circ g') \times 1_R)) = \delta$. Hence b is fully faithful. □

Proposition 4.15. *Let $\mathfrak{X}_0$ be a CFG/$\mathcal{S}$. Let $\mathfrak{X}$ and $\mathfrak{X}'$ be stackifications of $\mathfrak{X}$, by $b\colon \mathfrak{X}_0 \to \mathfrak{X}$ and $b'\colon \mathfrak{X}_0 \to \mathfrak{X}'$. Then there exists an isomorphism $f\colon \mathfrak{X} \to \mathfrak{X}'$ and a 2-isomorphism $f \circ b \Rightarrow b'$. If $g\colon \mathfrak{X} \to \mathfrak{X}'$ is another isomorphism, with $g \circ b \Rightarrow b'$, then there exists a unique 2-isomorphism $f \Rightarrow g$ such that the composite 2-isomorphism $f \circ b \Rightarrow g \circ b \Rightarrow g'$ is equal to the given 2-isomorphism $f \circ b \Rightarrow b'$.*

Proof. Taking $\mathfrak{Y}$ to be $\mathfrak{X}'$ in Definition 4.12, we have an equivalence of categories $\mathrm{HOM}(\mathfrak{X}, \mathfrak{X}') \to \mathrm{HOM}(\mathfrak{X}_0, \mathfrak{X}')$ induced by composition with b. In particular, there exists a morphism $f\colon \mathfrak{X} \to \mathfrak{X}'$ and a 2-isomorphism $f \circ b \Rightarrow b'$. Similarly, there is a morphism $f'\colon \mathfrak{X}' \to \mathfrak{X}$ and a 2-isomorphism $f' \circ b' \Rightarrow b$. Taking $\mathfrak{Y}$ to be $\mathfrak{X}$ in Definition 4.12, now, means that $\mathrm{HOM}(\mathfrak{X}, \mathfrak{X}) \to \mathrm{HOM}(\mathfrak{X}_0, \mathfrak{X})$ is an equivalence of categories. The objects $f' \circ f$ and $1_{\mathfrak{X}}$ of $\mathrm{HOM}(\mathfrak{X}, \mathfrak{X})$ map to isomorphic objects of $\mathrm{HOM}(\mathfrak{X}_0, \mathfrak{X})$, since $f' \circ f \circ b$ is 2-isomorphic to $f' \circ b'$, which in turn is 2-isomorphic to b. So there exists a 2-isomorphism $f' \circ f \Rightarrow 1_{\mathfrak{X}}$. Similarly, there exists a 2-isomorphism $f \circ f' \Rightarrow 1_{\mathfrak{X}'}$. This implies that f is an isomorphism (see the remark at the end of Section 2.3). If $g\colon \mathfrak{X} \to \mathfrak{X}'$ is another morphism, with an 2-isomorphism $g \circ b \Rightarrow b'$, then composing the 2-isomorphisms we have $f \circ b \Rightarrow g \circ b$, and this composite 2-isomorphism must, by full faithfulness of composition with b, come from a unique 2-isomorphism $f \Rightarrow g$. $\square$

Corollary 4.16. *Let $b\colon \mathfrak{X}_0 \to \mathfrak{X}$ be a stackification. Let $f_0\colon \mathfrak{X}_0 \to \mathfrak{Y}$ be a morphism to a stack $\mathfrak{Y}$, and let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism, such that $f \circ b$ is 2-isomorphic to f_0. If f_0 is fully faithful (as a functor) and satisfies the property that every object x of $\mathfrak{Y}$ over T in $\mathcal{S}$ admits morphisms $f_0(t_\alpha) \to x$ over $T_\alpha \to T$ in $\mathfrak{Y}$ for some covering family $\{T_\alpha \to T\}$, then f is an isomorphism.*

Remark 4.17. Stackification is a sort of analogue of sheafification. Sheafification transforms a presheaf into a sheaf, which is characterized by a universal property up to canonical isomorphism. The universal property of Definition 4.12 characterizes the stackification — not up to an isomorphism of categories (this would be to strong) — rather up to an isomorphism of CFGs which is unique up to a canonical 2-isomorphism. That is the content of Proposition 4.15.

An important application of the stackification property is a sort of functoriality, namely that a morphism of groupoid schemes from $R' \rightrightarrows U'$ to $R \rightrightarrows U$ determines a morphism of stacks of torsors $[R' \rightrightarrows U'] \to [R \rightrightarrows U]$ (at least, up to canonical 2-isomorphism). In the previous chapter, we saw how to obtain a morphism of prestacks $[R' \rightrightarrows U']^{\mathrm{pre}} \to [R \rightrightarrows U]^{\mathrm{pre}}$ (in fact, Example 3.14 tells us precisely that the morphisms of prestacks correspond to morphisms of groupoid schemes, and that 2-morphisms between these morphisms correspond to maps $U' \to R$). To get a morphism of stacks, however, requires additional discussion.

Let (ϕ, Φ) be a morphism of groupoid schemes from $R' \rightrightarrows U'$ to $R \rightrightarrows U$. We first remark that we do not *directly* get a morphism of stacks of torsors, in general. Indeed, for a groupoid of the form $G \rightrightarrows \Lambda$ the associated stack is equivalent to BG; morphisms of such groupoids are group homomorphisms, and we have seen that the passage from a group homomorphism to a morphism of stacks requires a descent-based construction (Example 2.9(2) and Remark 2.17). This is also why the best we can do is to get a morphism of stacks that is defined up to canonical 2-isomorphism.

The construction of the morphism of stacks corresponding to a morphism of groupoid schemes appears in the proof of the next proposition. Here is an informal summary. A trivial $(R' \rightrightarrows U')$-groupoid over T (corresponding to some $T \to U$) is sent to a trivial $(R \rightrightarrows U)$-groupoid: we compose with the given $U' \to U$ and then form the associated trivial torsor as displayed in (1). A general $(R' \rightrightarrows U')$-torsor is locally trivial; we can choose a trivalizing (étale) cover $T' \to T$ and associate a descent datum (the trivial torsor corresponding to $T' \to U'$ with a gluing map given by some

$T' \times_T T' \to R'$). Applying the given maps Φ and ϕ we obtain similar data with respect to R and U. This corresponds to descent data for an $(R \rightrightarrows U)$-torsor. By effective descent for $(R \rightrightarrows U)$-torsors we obtain an $(R \rightrightarrows U)$-torsor (defined up to canonical isomorphism) over T. (It is a good exercise to correlate this informal description with the formal proof of the proposition, which is phrased using the language of stacks and stackification.)

PROPOSITION 4.18. *Let (ϕ, Φ) be a morphism of groupoid schemes from $R' \rightrightarrows U'$ to $R \rightrightarrows U$. Assume that both the source and target groupoid schemes have the property that the relative diagonals are quasi-affine. Then there is an induced morphism of stacks*

$$[R' \rightrightarrows U'] \to [R \rightrightarrows U],$$

defined up to canonical 2-isomorphism.

PROOF. Composing the corresponding morphism of prestacks $[R' \rightrightarrows U']^{\mathrm{pre}} \to [R \rightrightarrows U]^{\mathrm{pre}}$ with the stackification morphism of $R \rightrightarrows U$, we have a morphism

$$[R' \rightrightarrows U']^{\mathrm{pre}} \to [R \rightrightarrows U]. \tag{3}$$

By the universal property of the stackification of $R' \rightrightarrows U'$, there is a morphism $[R' \rightrightarrows U'] \to [R \rightrightarrows U]$ whose composite with the stackification $[R' \rightrightarrows U']^{\mathrm{pre}} \to [R' \rightrightarrows U']$ is 2-isomorphic to the morphism (3). The resulting morphism is unique up to canonical 2-isomorphism. $\square$

We conclude this section with a pair of results that complete the dictionary between stacks as categories and their groupoid presentations. If we start with a stack $\mathfrak{X}$ and obtain from it, by Proposition 3.5, a groupoid scheme $R \rightrightarrows U$, then we have the following criterion to have $\mathfrak{X} \cong [R \rightrightarrows U]$.

PROPOSITION 4.19. *Let $\mathfrak{X}$ be a stack, U a scheme, and u an object of $\mathfrak{X}_U$, and $\mathfrak{Sym}_{\mathfrak{X}}(u,u) \cong \underline{R}$ an isomorphism, for some scheme R. Let $R \rightrightarrows U$ be the associated symmetry groupoid, and assume that its relative diagonal $R \to U \times U$ is quasi-affine. If, for every object t of $\mathfrak{X}$ (over a scheme T) there exists a covering family $\{\varphi_i \colon T_i \to T\}$, such that $\varphi_i^* t$ admits a morphism in $\mathfrak{X}$ to u, for every i, then we have $\mathfrak{X} \cong [R \rightrightarrows U]$.*

PROOF. There is an evident morphism

$$F_0 \colon [R \rightrightarrows U]^{\mathrm{pre}} \to \mathfrak{X},$$

that we obtain as follows. We have $\underline{U} \to \mathfrak{X}$, corresponding to the object u of $\mathfrak{X}$ and associating, to any scheme S with morphism to $g \colon S \to U$, a chosen pullback g^*u of u. Let S' be another scheme, with $f \colon S' \to S$ and $g' \colon S' \to U$. There is a pullback morphism $(g \circ f)^*u \to g^*u$ (the unique morphism, over f, whose composite with the chosen $g^*u \to u$ is the chosen $(g \circ f)^*u \to u$). If $\gamma \colon S' \to R$ is a morphism in $[R \rightrightarrows U]^{\mathrm{pre}}$ over f (from the object g' to the object g), then we apply the given isomorphism

$$\underline{R} \to \underline{U} \times_{\mathfrak{X}} \underline{U} \tag{4}$$

and obtain the pair of objects (of $\underline{U}$) $s \circ \gamma$ (which is g') and $t \circ \gamma$ (which is $g \circ f$) and an isomorphism $g'^*u \to (g \circ f)^*u$ in $\mathfrak{X}$ (over $1_{S'}$). We declare $F_0(\gamma)$ to be the composite

$$g'^*u \to (g \circ f)^*u \to g^*u$$

with the pullback morphism.

In fact, composing with the pullback morphism gives a bijection between morphisms in $\mathfrak{X}$ over f from g'^*u to g^*u and morphisms in $\mathfrak{X}_{S'}$ from g'^*u to $(g \circ f)^*u$. Since (4) is an isomorphism of CFGs, these correspond bijectively with morphisms of schemes $S' \to R$ whose composite with s is g' and whose composite with t is $g \circ f$. These are precisely the morphisms in $[R \rightrightarrows U]^{\mathrm{pre}}$, and hence F_0 is fully faithful, as a functor.

The remaining hypothesis guarantees that F_0 satisfies the conditions of Corollary 4.16. Hence if $F\colon [R \rightrightarrows U] \to \mathfrak{X}$ denotes an associated morphism of stacks (by the stackification property), then it follows that F is an isomorphism. $\square$

Proposition 4.20. *Let $R \rightrightarrows U$ be a groupoid scheme with quasi-affine relative diagonal, and let $\mathfrak{X} := [R \rightrightarrows U]$ be the associated stack of torsors. Denote by u the object of $[R \rightrightarrows U]$ over U given by the $(R \rightrightarrows U)$-groupoid $s\colon R \to U$ of Example 4.7. (This is the trivial $(R \rightrightarrows U)$-groupoid associated with the identity morphism 1_U.) Then we have an isomorphism $\mathfrak{Sym}_{\mathfrak{X}}(u,u) \cong \underline{R}$, and the groupoid scheme that we recover from this isomorphism by Prop 3.5 is precisely the given groupoid $R \rightrightarrows U$.*

Proof. Notice that the morphism $\underline{U} \to [R \rightrightarrows U]$ that is associated with u factors through $[R \rightrightarrows U]^{\mathrm{pre}}$. Since the stackification morphism $[R \rightrightarrows U]^{\mathrm{pre}} \to [R \rightrightarrows U]$ is a fully faithful functor, it follows that the category $\mathfrak{Sym}_{\mathfrak{X}}(u,u)$ is identified (by an isomorphism of categories) with the fiber product

$$\underline{U} \times_{[R \rightrightarrows U]^{\mathrm{pre}}} \underline{U}.$$

Now an object of this fiber product (over a scheme T) consists of a pair of morphisms h, $h'\colon T \to U$ and a morphism $r\colon T \to R$ whose composite with s is h and whose composite with t is h'. The morphisms h and h' are completely determined by r, so we see that $\underline{U} \times_{[R \rightrightarrows U]^{\mathrm{pre}}} \underline{U} \cong \underline{R}$. It is also clear from this description that groupoid we obtain by applying Prop 3.5 has structure maps s, $t\colon R \to U$.

It remains only to check that the other maps of the groupoid structure that we obtain agree with the given ones. We do this for the multiplication map, leaving the others for the reader to check. Consider morphisms h, h', $h''\colon T \to U$ and r, $r'\colon T \to R$, satisfying

$$s \circ r = h, \qquad t \circ r = h' = s \circ r', \qquad t \circ r' = h''$$

(an object of $\underline{U} \times_{[R \rightrightarrows U]^{\mathrm{pre}}} \underline{U} \times_{[R \rightrightarrows U]^{\mathrm{pre}}} \underline{U} \cong \underline{R}\ {}_t\!\times_s \underline{R}$). The multiplication map arising from Proposition 3.5 produces the composite (in $[R \rightrightarrows U]^{\mathrm{pre}}$) of r and r'. Looking at the description in Definition 3.11, the composite is the map $m(r,r')\colon T \to R$. So the pair of composable arrows r and r' are sent to $m(r,r')$, hence the multiplication map of the resulting groupoid structure is $m\colon R\ {}_t\!\times_s R \to R$. $\square$

5. Stack realizations of groupoid constructions

We saw some examples of concrete constructions that can be made with algebraic groupoids in Section 3.3. Now we can show that each construction has a consequence for the associated stacks of torsors.

All groupoid schemes in this section are assumed to have quasi-affine relative diagonal.

Example 4.21. Given groupoid schemes $R \rightrightarrows U$, $R' \rightrightarrows U'$, and $R'' \rightrightarrows U''$, and morphisms of groupoid schemes $(R' \rightrightarrows U') \to (R \rightrightarrows U)$ and $(R'' \rightrightarrows U'') \to (R \rightrightarrows U)$, then we have

$$[R' \rightrightarrows U'] \times_{[R \rightrightarrows U]} [R'' \rightrightarrows U''] \cong [R' \times_U R \times_U R'' \rightrightarrows U' \times_U R \times_U U''].$$

Indeed, we have a morphism from the fiber product of the associated prestacks to $[R' \rightrightarrows U'] \times_{[R \rightrightarrows U]} [R'' \rightrightarrows U'']$ (by the universal property of the fiber product). This is readily seen to be fully faithful. Moreover any object of $[R' \rightrightarrows U'] \times_{[R \rightrightarrows U]} [R'' \rightrightarrows U'']$ becomes, after étale pullback, isomorphic to an object in the image of this morphism. So the criterion of Proposition 4.13 is satisfied. Combining the isomorphism of Example 3.13 with Proposition 4.15, we get the desired isomorphism of stacks.

Example 4.22. Given groupoid schemes $R' \rightrightarrows U'$ and $R \rightrightarrows U$, we try to describe the category $\mathrm{HOM}([R' \rightrightarrows U'], [R \rightrightarrows U])$. We know that a morphism of groupoid schemes determines (up to canonical 2-isomorphism) a morphism $[R' \rightrightarrows U'] \to [R \rightrightarrows U]$. Notice that there can be morphisms between the stacks, not corresponding directly to any morphism of groupoid scheme. Suppose, however, that (ϕ, Φ) and $(\tilde{\phi}, \widetilde{\Phi})$ are morphisms of groupoid schemes, with respective corresponding morphisms

$$f, \tilde{f} \colon [R' \rightrightarrows U'] \to [R \rightrightarrows U].$$

Then the 2-isomorphisms $f \Rightarrow \tilde{f}$ will be in canonical bijection with the morphisms of schemes $\gamma \colon U' \to R$ satisfying $s \circ \gamma = \phi$, $t \circ \gamma = \tilde{\phi}$, and $m(\gamma \circ s, \widetilde{\Phi}) = m(\Phi, \gamma \circ t)$.

To justify this, we notice that composition with the stackification morphism of $[R \rightrightarrows U]^{\mathrm{pre}}$ is a fully faithful functor

$$\mathrm{HOM}([R' \rightrightarrows U']^{\mathrm{pre}}, [R \rightrightarrows U]^{\mathrm{pre}}) \to \mathrm{HOM}([R' \rightrightarrows U']^{\mathrm{pre}}, [R \rightrightarrows U]).$$

The category on the right is equivalent to the category $\mathrm{HOM}([R' \rightrightarrows U'], [R \rightrightarrows U])$ by the stackification property. So there is a fully faithful functor

$$\mathrm{HOM}([R' \rightrightarrows U']^{\mathrm{pre}}, [R \rightrightarrows U]^{\mathrm{pre}}) \to \mathrm{HOM}([R' \rightrightarrows U'], [R \rightrightarrows U]).$$

That means that we can appeal to Example 3.14, where the morphisms in the HOM-category of prestacks are described in terms of morphisms of schemes $U' \to R$.

Remark 4.23. For the sake of simplicity, the discussions of groupoids have focused only on groupoid schemes and morphisms of groupoid schemes. However, groupoid schemes are part of a richer structure, namely a *2-category*, where the 2-morphisms from (ϕ, Φ) to $(\tilde{\phi}, \widetilde{\Phi})$ are precisely the morphisms of schemes $\gamma \colon U' \to R$ satisfying $s \circ \gamma = \phi$, $t\gamma = \tilde{\phi}$, and $m(\gamma \circ s, \widetilde{\Phi}) = m(\Phi, \gamma \circ t)$. We do not need this extra level of formalism (which involves composition operations involving 2-morphisms, and a host of axioms); the interested reader can find a discussion of 2-categories in Appendix B, where the 2-category of groupoid schemes (more generally, $\mathcal{S}$-groupoids for general $\mathcal{S}$) is introduced in Exercise B.25. Comparing with Example 3.14, in fact, the 2-category of groupoid schemes is faithfully represented by the CFGs $[R \rightrightarrows U]^{\mathrm{pre}}$. (CFGs, with their morphisms and 2-morphisms, also form a 2-category, and the 2-category of $\mathcal{S}$-groupoids is 2-equivalent to a full sub-2-category of CFGs/$\mathcal{S}$.)

Example 4.24. Let a groupoid scheme $R \rightrightarrows U$ be given. If $R' \rightrightarrows U'$ is a groupoid scheme with morphism of groupoid schemes (ϕ, Φ) to $R \rightrightarrows U$ satisfying Condition 1.3(i)–(ii) (this means R' is isomorphic to $R \times_{U \times U} (U' \times U')$ and $U' \times_U R \to U$ admits sections étale locally) then we claim that the corresponding

$$f\colon [R' \rightrightarrows U'] \to [R \rightrightarrows U]$$

is an isomorphism. If we follow the steps of the previous example, we see that we take f_0 to be the composite $[R' \rightrightarrows U']^{\mathrm{pre}} \to [R \rightrightarrows U]^{\mathrm{pre}} \to [R \rightrightarrows U]$ and then f arises from f_0 by the stackification property for $[R' \rightrightarrows U']$. As a functor, f_0 is fully faithful, since it is a composite of fully faithful functors. If we can show that every $(R \rightrightarrows U)$-torsor becomes, after pullback to an étale cover, isomorphic to something in the image of f_0, then it will follow by Corollary 4.16 that f is an isomorphism. An $(R \rightrightarrows U)$-torsor is locally trivial, hence locally isomorphic to the image of an object of $[R \rightrightarrows U]^{\mathrm{pre}}$. This in turn is locally isomorphic to the image of an object of $[R' \rightrightarrows U']^{\mathrm{pre}}$. For this last step, start with $T \to U$. By Condition 1.3(ii), we have a covering map $T' := U' \times_U R \times_U T \to T$. There are morphisms $\bar{s}, \bar{t}\colon T' \to U$ corresponding to the maps s, t respectively from the middle factor R. They are isomorphic objects of $[R \rightrightarrows U]^{\mathrm{pre}}$ over T'. The latter is the pullback of the given object over T, while the former is in the image of the morphism $[R' \rightrightarrows U']^{\mathrm{pre}} \to [R \rightrightarrows U]^{\mathrm{pre}}$.

In particular, for X a scheme we have $[X \rightrightarrows X] \cong \underline{X}$ (easy), so if $U \to X$ is a covering map and $R = U \times_X U$ then $[R \rightrightarrows U] \cong \underline{X}$.

Remark 4.25. It is possible to take a different point of view on the stackification of $[R \rightrightarrows U]^{\mathrm{pre}}$. As we observed, a typical descent datum would give rise to morphism of groupoid schemes from a banal groupoid $T'' \rightrightarrows T'$ to $R \rightrightarrows U$. So we can consider the category where an object over T is a banal groupoid over an arbitrary covering map $f\colon T' \to T$, together with a morphism of groupoid schemes from $T'' \rightrightarrows T'$ to $R \rightrightarrows U$. Thinking of T as represented by the groupoid scheme $T \rightrightarrows T$, and using the notion of equivalence of groupoid schemes of Condition 1.3(i)–(ii), we could declare that a morphism of groupoid schemes, in a new and fancier sense, from $R' \rightrightarrows U'$ to $R \rightrightarrows U$ will consist of a pair of (traditional) morphisms of groupoid schemes

$$\begin{tikzcd} R' \arrow[d, shift left] \arrow[d, shift right] & R'' \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & R \arrow[d, shift left] \arrow[d, shift right] \\ U' & U'' \arrow[l] \arrow[r] & U \end{tikzcd}$$

in which the left-hand morphism is an equivalence of groupoids. To make this work, we really must be working in the 2-category of groupoid schemes. Further, we would require a technique called *localization in a 2-category*. The resulting structure will be one in which the category of morphisms from $T \rightrightarrows T$ to $R \rightrightarrows U$ is equivalent to the fiber $[R \rightrightarrows U]_T$ of our category of $(R \rightrightarrows U)$-torsors. This is a way to give sense to the claim that one finds, e.g. in [**89**], that $[R \rightrightarrows U]_T$ will be a sort of direct limit over covering maps $T' \to T$ of the category of morphisms of groupoid schemes from $T'' \rightrightarrows T'$ to $R \rightrightarrows U$. More generally, the category of morphisms from $R' \rightrightarrows U'$ to $R \rightrightarrows U$, after localization, will be equivalent to our category $\mathrm{HOM}([R' \rightrightarrows U'], [R \rightrightarrows U])$. (All these statements are subject to some additional hypothesis on the groupoid schemes, which amounts to saying that they are the groupoid schemes of algebraic stacks.)

Answers to Exercises

4.1. The CFG in (a) is a stack; we can apply descent for projective morphisms taking as relatively ample invertible sheaf the relative anticanonical sheaf a smooth family of genus 0 curves. Any finite morphism is affine, so (b) is a stack. In (c) the prestack axiom would say that two families of elliptic curves which are locally isomorphic must be isomorphic; the existence of isotrivial families of elliptic curves means that this CFG is not even a prestack. The CFG in (d) is a prestack by the usual argument, descent for morphisms to a target scheme, but it is not a stack: a conic over a non-algebraically closed field (of characteristic $\neq 2$) with no rational points is not the projectivization of a vector bundle, but it becomes isomorphic to $\mathbb{P}^1$ after a quadratic field extension.

4.2. We refer to the diagrams of Axiom (2) for groupoid scheme, which are cartesian by Exercise 3.1(d). According to the left-hand diagram, $R\,{}_t{\times}_s R$ is identified, by (pr_1, m), with the fiber product of $s\colon R \to U$ with itself. Hence pr_1, $m\colon R\,{}_t{\times}_s\, R \rightrightarrows R$ is a banal groupoid for s. The multiplication map $m'\colon (R\,{}_t{\times}_s\, R)\,{}_m{\times}_{\mathrm{pr}_1} (R\,{}_t{\times}_s\, R) \to R\,{}_t{\times}_s\, R$ of the algebraic groupoid structure is the unique map satisfying $\mathrm{pr}_1 \circ m' = \mathrm{pr}_1 \circ \mathrm{pr}_1$ and $m \circ m' = m \circ \mathrm{pr}_2$, and this must send $((\alpha, \beta), (m(\alpha, \beta), \gamma))$ to $(\alpha, m(\beta, \gamma))$. To see that (t, pr_2) is a morphism of groupoid schemes, we must check the compatibility conditions, e.g., $m(\mathrm{pr}_2 \times \mathrm{pr}_2) = \mathrm{pr}_2 \circ m'$ since both composites send $((\alpha, \beta), (m(\alpha, \beta), \gamma))$ to $m(\beta, \gamma)$. Finally, this is a square morphism: the relevant cartesian diagrams are the obvious one with $R\,{}_t{\times}_s\, R$ as the fiber product of t and s, and the right-hand diagram of Axiom (2).

4.3. Since $g \circ \phi \circ s = g \circ \phi t$, there is a unique $\Phi\colon R \to X''$ such that $p \circ \Phi = \phi \circ s$ and $q \circ \Phi = \phi \circ t$. The remaining identities required of a morphism of groupoid schemes are identities of morphisms to X'', and it suffices to verify such an identity after composing with p and with q. For instance, we require $m' \circ (\Phi \times \Phi) = \Phi \circ m$, where m' denotes the multiplication map of the groupoid scheme $X'' \rightrightarrows X'$, and this holds since $p \circ m' \circ (\Phi \times \Phi) = p \circ \Phi \circ \mathrm{pr}_1 = \phi \circ s \circ \mathrm{pr}_1 = \phi \circ s \circ m = p \circ \Phi \circ m$, and similarly $q \circ m' \circ (\Phi \times \Phi) = q \circ \Phi \circ m$.

4.4. Let $s_{\alpha\gamma\varepsilon\lambda} \to s_{\alpha\gamma\varepsilon}$ be a morphism over $S_{\alpha\gamma\varepsilon\lambda} \to S_{\alpha\gamma\varepsilon}$, so that the composite $s_{\alpha\gamma\varepsilon\lambda} \to s_{\alpha\gamma\varepsilon} \to s_\gamma$ lies over $S_{\alpha\gamma\varepsilon\lambda} \to S_\gamma$. Let $s_{\alpha\gamma\varepsilon} \to t_\alpha$ be the morphism that is stipulated in the proof of the proposition, i.e., such that $b(s_{\alpha\gamma\varepsilon}) \to b(t_\alpha) \to x$ is equal to the composite $b(s_{\alpha\gamma\varepsilon}) \to b(s_\gamma) \to u \to x$. Now the composite $b(s_{\alpha\gamma\varepsilon\lambda}) \to b(s_{\alpha\gamma\varepsilon}) \to b(t_\alpha) \to x$ is equal to the composite $b(s_{\alpha\gamma\varepsilon\lambda}) \to b(s_{\alpha\gamma\varepsilon}) \to b(s_\gamma) \to u \to x$. So, in the construction applied to the refined cover, we must take as morphism over $S_{\alpha\gamma\varepsilon\lambda} \to T_\alpha$ the composite $s_{\alpha\gamma\varepsilon\lambda} \to s_{\alpha\gamma\varepsilon} \to t_\alpha$. The proposition produces $v \to y$ such that the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to y$ is equal to $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(t_\alpha) \to y$. So the composite $f_0(s_{\alpha\gamma\varepsilon\lambda}) \to f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to y$ is equal to $f_0(s_{\alpha\gamma\varepsilon\lambda}) \to f_0(s_{\alpha\gamma\varepsilon}) \to f_0(t_\alpha) \to y$, and thus the same morphism $v \to y$ is the unique morphism dictated by Axiom 1 for the refined cover.