CHAPTER 3

Groupoids and Atlases

For an atlas for a stack, one has schemes R and U, with a morphism from R to $U \times U$ (not usually an embedding). Conditions are put on these data that, in case R is contained in $U \times U$, make R an equivalence relation. This is the notion of a *groupoid scheme*, which is the subject of this chapter (cf. [**30**]). These will be the *atlases* (or *groupoid presentations*) for stacks.

1. Groupoid schemes

A **groupoid scheme**, or **algebraic groupoid**,[1] consists of two schemes and five morphisms, satisfying several properties. One has a scheme U, a scheme R, two morphisms s and t from R to U, a morphism e from U to R, a morphism $m\colon R\ {}_t{\times}_s\, R \to R$ (where $R\ {}_t{\times}_s\, R$ denotes the fiber product $R \times_U R$ constructed from the two maps t and s), and a morphism $i\colon R \to R$, satisfying the five properties listed below.

All of this makes sense over an arbitrary base category $\mathcal{S}$, and then one defines a **groupoid object** in $\mathcal{S}$. A key example to keep in mind is a groupoid set. This notion coincides precisely with the notion of a small category in which all morphisms are isomorphisms. From this example one can in fact figure out what the axioms must be. Here are the axioms:

(1) The composites $U \xrightarrow{e} R \xrightarrow{s} U$ and $U \xrightarrow{e} R \xrightarrow{t} U$ are the identity maps on U:

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[dr, "1_U"'] & R \arrow[d, "s"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "e"] \arrow[dr, "1_U"'] & R \arrow[d, "t"] \\ & U \end{tikzcd}$$

(2) If pr_1 and pr_2 are the two projections from $R\ {}_t{\times}_s\, R$ to R, then $s \circ m = s \circ \mathrm{pr}_1$ and $t \circ m = t \circ \mathrm{pr}_2$:

$$\begin{tikzcd} R\ {}_t{\times}_s\, R \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[d, "s"] \\ R \arrow[r, "s"'] & U \end{tikzcd} \qquad \begin{tikzcd} R\ {}_t{\times}_s\, R \arrow[r, "\mathrm{pr}_2"] \arrow[d, "m"'] & R \arrow[d, "t"] \\ R \arrow[r, "t"'] & U \end{tikzcd}$$

[1] The second choice of "algebraic groupoid" compares nicely with "algebraic group", and is common in the literature. It does conflict with the use of groupoid for a kind of category. In category language, cf. [**65**], what we call a groupoid scheme is an "internal groupoid in the category of schemes".

(3) (Associativity) The two maps $m\circ(1_R\times m)$ and $m\circ(m\times 1_R)$ from $R\,{}_t{\times}_s\,R\,{}_t{\times}_s\,R$ to R are equal:

$$\begin{tikzcd} R\,{}_t{\times}_s\,R\,{}_t{\times}_s\,R \arrow[r, "1_R\times m"] \arrow[d, "m\times 1_R"'] & R\,{}_t{\times}_s\,R \arrow[d, "m"] \\ R\,{}_t{\times}_s\,R \arrow[r, "m"'] & R \end{tikzcd}$$

(4) (Unit) The two maps $m\circ(e\circ s, 1_R)$ and $m\circ(1_R, e\circ t)$ from R to R are equal to the identity on R:

$$\begin{tikzcd} R \arrow[r, "(e\circ s,1_R)"] \arrow[dr, "1_R"'] & R\,{}_t{\times}_s\,R \arrow[d, "m"] \\ & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "(1_R,e\circ t)"] \arrow[dr, "1_R"'] & R\,{}_t{\times}_s\,R \arrow[d, "m"] \\ & R \end{tikzcd}$$

(5) (Inverses) $i\circ i = 1_R$, $s\circ i = t$ and (therefore) $t\circ i = s$, and $m\circ(1_R, i) = e\circ s$ and $m\circ(i, 1_R) = e\circ t$:

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "1_R"'] & R \arrow[d, "i"] \\ & R \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[d, "t"] \\ & R \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "(1_R,i)"] \arrow[d, "s"'] & R\,{}_t{\times}_s\,R \arrow[d, "m"] \\ U \arrow[r, "e"'] & R \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "(i,1_R)"] \arrow[d, "t"'] & R\,{}_t{\times}_s\,R \arrow[d, "m"] \\ U \arrow[r, "e"'] & R \end{tikzcd}$$

Note by (1) that s and t are always surjective. These axioms have some redundancy, but others could be added. For example:

Exercise 3.1. (a) Show that $i\circ e = e$. (b) Show that $m\circ(e,e) = e$. (c) Show that $m\circ(i\circ \mathrm{pr}_2, i\circ \mathrm{pr}_1) = i\circ m$. (d) Show that the diagrams of axiom (2) are cartesian.

Example 3.1. Any morphism $U\to X$ of schemes determines a groupoid scheme. For this, take $R = U\times_X U$, with s and t the two projections, e the diagonal map, and i the map switching the two factors. Identifying $R\,{}_t{\times}_s\,R$ with $U\times_X U\times_X U$, the map m is the projection onto the first and third factors.

Example 3.2. If $U = \Lambda$ (the base scheme) then the axioms for a groupoid scheme reduce to axioms for R to be a group scheme over Λ (with multiplication $m\colon R\times R\to R$, identity section $e\colon \Lambda\to R$, and inverse map $i\colon R\to R$).

Example 3.3. An important example of a groupoid scheme arises whenever an algebraic group G acts on the right on a scheme U. Set $R = U\times G$, and let $s\colon U\times G\to U$ be projection and $t\colon U\times G\to U$ the action. The map $e\colon U\to U\times G$ takes u to (u, e_G), where e_G is the identity element of G. The map i takes (u,g) to $(u\cdot g, g^{-1})$, and

$$m\big((u,g),(u\cdot g,h)\big) = (u, gh).$$

We may identify $R\,{}_t{\times}_s\,R$ with $U\times G\times G$ by the map $\big((u,g),(u\cdot g,h)\big)\mapsto(u,g,h)$. Under this identification, m becomes the map $(u,g,h)\mapsto(u,gh)$.

EXAMPLE 3.3′. For a left action, we have a groupoid scheme with $R = G \times U$, s the projection, t the action, $e(u) = (e_G, u)$, $i(g,u) = (g^{-1}, g \cdot u)$, and $m\big((g,u),(h, g\cdot u)\big) = (hg, u)$.

A groupoid scheme can be denoted (U, R, s, t, m, e, i), or more simply $R \rightrightarrows U$. We call the groupoid scheme an **étale groupoid scheme** if the two morphisms s and t are étale. Similar terminology is used for other adjectives such as smooth or flat. The groupoid scheme for $\mathcal{M}_{1,1}$ in Example 3.10 is an étale groupoid scheme. The groupoids $X \times G \rightrightarrows X$ and $G \times X \rightrightarrows X$ (Examples 3.9 and 3.9′) are étale groupoids when G is étale over the base field (e.g., a finite group), and are smooth groupoids when G is smooth (e.g., an algebraic group over a field).

A **morphism** of groupoid schemes from $(U', R', s', t', m', e', i')$ to (U, R, s, t, m, e, i) is a pair (ϕ, Φ), where $\phi\colon U' \to U$ and $\Phi\colon R' \to R$ are morphisms of schemes. These are required to be compatible with the structure morphisms defining each groupoid scheme, in the obvious sense: $s \circ \Phi = \phi \circ s'$, $t \circ \Phi = \phi \circ t'$, $e \circ \phi = \Phi \circ e'$, $m \circ (\Phi \times \Phi) = \Phi \circ m'$, and $i \circ \Phi = \Phi \circ i'$.

EXAMPLE 3.4. If G acts on U, and H acts on V, and $\theta\colon G \to H$ is a homomorphism of algebraic groups, and $\phi\colon U \to V$ is an equivariant map (so $\phi(u,g) = \phi(u) \cdot \theta(g)$ for $u \in U$ and $g \in G$), this determines a morphism (ϕ, Φ) from the groupoid scheme $U \times G \rightrightarrows U$ to $V \times H \rightrightarrows V$, with $\Phi(u,g) = (\phi(u), \theta(g))$.

2. Groupoids and CFGs

It will be possible to go back and forth between algebraic groupoids and CFGs. An algebraic groupoid will describe a CFG, much the way that a scheme can be described by patching. But this goes via a process that requires several steps; these will be presented over the course of this chapter and the next chapter. We start with the easier direction, that of producing an algebraic groupoid from a CFG.

PROPOSITION 3.5. *Let $\mathfrak{X}$ be a CFG, U a scheme, and $u\colon \underline{U} \to \mathfrak{X}$ a morphism. Assume given an isomorphism between $\underline{U} \times_{\mathfrak{X}} \underline{U}$ and $\underline{R}$ for some scheme R. Then R and U form a groupoid scheme, with the pair of projection maps*

$$s, t\colon \underline{R} \cong \underline{U} \times_{\mathfrak{X}} \underline{U} \rightrightarrows \underline{U}.$$

and the following additional maps:

(i) *$e\colon U \to R$ is the composite $\underline{U} \to \underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}$, where the first map sends h to $(h, h, 1_{u(h)})$,*

(ii) *m is the map from $\underline{U} \times_{\mathfrak{X}} \underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}\ {}_t\!\times_{\underline{U},s} \underline{R}$ to $\underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}$ given by*

$$(h, h', h'', \varphi, \varphi') \mapsto (h, h'', \varphi' \circ \varphi),$$

(iii) *$i\colon \underline{R} \cong \underline{U} \times_{\mathfrak{X}} \underline{U} \to \underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}$, where $(h, h', \varphi) \mapsto (h', h, \varphi^{-1})$.*

In the definition of m, notice that we make use of the isomorphism $\underline{U} \times_{\mathfrak{X}} \underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}\ {}_t\!\times_{\underline{U},s} \underline{R}$ of Example 2.27. We also repeatedly use the correspondence between morphisms of schemes and morphisms of the associated CFGs (Example 2.9(1)).

PROOF. We have to verify the axioms. We verify the Associativity axiom (3) and leave the verification of the other axioms to the reader. Let

$$(h, h', h'', h''', \varphi, \varphi', \varphi'')$$

be an object of $\underline{U} \times_{\mathfrak{X}} \underline{U} \times_{\mathfrak{X}} \underline{U} \times_{\mathfrak{X}} \underline{U}$, which is identified with $\underline{R}\ {}_t\times_s \underline{R}\ {}_t\times_s \underline{R}$. If we apply $m \times 1_R$, we get $(h, h'', h''', \varphi' \circ \varphi, \varphi'')$. Applying m produces $(h, h''', \varphi'' \circ (\varphi' \circ \varphi))$. If, intead, we apply $1_R \times m$ and then m, we get $(h, h''', (\varphi'' \circ \varphi') \circ \varphi)$. Since composition of morphisms in the category $\mathfrak{X}$ is associative, we have $\varphi'' \circ (\varphi' \circ \varphi) = (\varphi'' \circ \varphi') \circ \varphi$. So, the two maps $m \circ (1_R \times m)$ and $m \circ (m \times 1_R)$ are equal. $\square$

According to Proposition 2.20, to specify the morphism $u \colon \underline{U} \to \mathfrak{X}$ is equivalent to specifying an object of $\mathfrak{X}$ over U. We are adopting the notational convention to use the same symbol for both the object and the morphism; this means that for a given morphism $f \colon S \to U$ of schemes, $u(S, f)$ will be the (chosen) pullback $f^*(u)$. Now Proposition 3.5 can be interpreted as saying that an object u of $\mathcal{X}$ over a scheme U determines an algebraic groupoid, provided that the corresponding fiber product of $\underline{U}$ with itself over $\mathfrak{X}$ is isomorphic to a scheme.

Fiber products of schemes over a target CFG will occur so frequently, that they deserve a special notation. We introduce this now.

DEFINITION 3.6. Let $\mathfrak{X}$ be a CFG over a base category $\mathcal{S}$ of schemes. Let U and V be schemes, and let u and v be objects of $\mathfrak{X}$ over U and over V, respectively. These determine morphisms (which are unique up to canonical 2-isomorphisms) $u \colon \underline{U} \to \mathfrak{X}$ and $v \colon \underline{V} \to \mathfrak{X}$. The **symmetry CFG** of u and v is the fiber product $\underline{U}\ {}_u\times_{\mathfrak{X},v} \underline{V}$. It will be denoted $\mathfrak{Sym}_{\mathfrak{X}}(u, v)$.

By Remark 2.26, the CFG $\mathfrak{Sym}_{\mathfrak{X}}(u, v)$ has sets as fibers. Concretely, the fiber over S is the set of triples (f, g, φ) where $f \colon S \to U$ and $g \colon S \to V$ are morphisms and $\varphi \colon f^*(u) \to g^*(v)$ is an isomorphism in $\mathfrak{X}_S$. To say that $\mathfrak{Sym}_{\mathfrak{X}}(u, v)$ is isomorphic to $\underline{T}$ for some scheme T is to say that the set of triples (f, g, φ) is naturally isomorphic to the set of morphisms $S \to T$. This condition only depends on the isomorphism class of u and the isomorphism class of v.

According to Proposition 3.5, now, if $\mathfrak{Sym}_{\mathfrak{X}}(u, u)$ is isomorphic to a scheme, then it determines an algebraic groupoid.

DEFINITION 3.7. Let $\mathfrak{X}$ be a CFG, U a scheme, and u an object of $\mathfrak{X}_U$. If there is a scheme R and an isomorphism $\mathfrak{Sym}_{\mathfrak{X}}(u, u) \cong \underline{R}$, then the groupoid $R \rightrightarrows U$ of Proposition 3.5 will be called the **symmetry groupoid** of u.

The notation $\mathfrak{Sym}_{\mathfrak{X}}(u, u) \rightrightarrows U$ could be used to denote the symmetry groupoid, although we will tend to avoid doing this, because it hides the fact that there is a nontrivial hypothesis in Proposition 2.20, that the fiber product of $\underline{U}$ with itself over $\mathfrak{X}$ should be isomorphic to a scheme.

The notions of symmetry CFG and symmetry groupoid also make sense over an arbitrary base category $\mathcal{S}$. The symmetry groupoid of u, when it exists, will be a groupoid object in $\mathcal{S}$.

Here are some examples of symmetry groupoids.

Example 3.8. Suppose $\mathfrak{X} = \underline{X}$. Let $f\colon U \to X$ be a morphism, determining (by composition with f) a morphism $\underline{f}\colon \underline{U} \to \underline{X}$. The fiber product $\underline{U} \times_{\underline{X}} \underline{U}$ is isomorphic to $\underline{U \times_X U}$ by Example 2.25(1). So, the corresponding symmetry groupoid is the algebraic groupoid of Example 3.1.

Example 3.9. For $\mathfrak{X} = [X/G]$, we take $U = X$ with the trivial G-torsor $X \times G$ and action map $X \times G \to X$. The corresponding $f\colon \underline{X} \to [X/G]$ maps $h\colon S \to X$ to the torsor $S \times G \to S$ with equivariant map $S \times G \to X$, $(s,g) \mapsto h(s)\cdot g$ (cf. Example 2.9(4)). So an object of $\underline{X} \times_{[X/G]} \underline{X}$ consists of h, $h'\colon S \to X$, and G-equivariant $S \times G \to S \times G$, $(s,g) \mapsto (s, \varphi(s)g)$ (for some $\varphi\colon S \to G$) such that the diagram

$$\begin{tikzcd} S \times G \arrow[rr, "h(s)\cdot g"] \arrow[dr, "(s,\varphi(s)g)"'] & & X \\ & S \times G \arrow[ur, "h'(s)\cdot g"'] & \end{tikzcd}$$

commutes, i.e., $h'(s)\cdot\varphi(s) = h(s)$. In other words, $h'(s) = h(s)\cdot\varphi(s)^{-1}$. So, we identify $\underline{X} \times_{[X/G]} \underline{X}$ with $\underline{X} \times \underline{G}$ so that h, h', and $S \times G \xrightarrow{\sim} S \times G$ as above are sent to (h, φ^{-1}). Then, s is the first projection and t is the group action. We have $e = (1_X, e_G)$. To compute m, say $((h,\varphi^{-1}), (h', \varphi'^{-1}))$ is an object of $\underline{X} \times_{[X/G]} \underline{X} \times_{[X/G]} \underline{X}$. The composite isomorphism $S \times G \to S \times G$ is $(s,g) \mapsto (s, \varphi'(s)\varphi(s)g)$. The inverse of $\varphi'(s)\varphi(s)$ is $\varphi(s)^{-1}\varphi'(s)^{-1}$, so m sends $((h,\varphi^{-1}), (h',\varphi'^{-1}))$ to $(h, \varphi^{-1}\varphi'^{-1})$. Similarly, i sends (h, φ^{-1}) to $(h\cdot\varphi^{-1}, \varphi)$. We have reproduced the algebraic groupoid of Example 3.3.

In particular, for $\mathfrak{X} = BG$ we can take $U = \Lambda$ and obtain the groupoid scheme $G \rightrightarrows \Lambda$ with $m\colon G \times G \to G$ the multiplication of G and $i\colon G \to G$ the inverse.

Example 3.9′. There is a similar story for $\mathfrak{X} = [G\backslash X]$, again with $U = X$. We take f to be the morphism $\underline{X} \to [G\backslash X]$ which sends $h\colon S \to X$ to $G \times S \to S$ and $G \times S \to X$, $(g,s) \mapsto g\cdot h(s)$. An object of $\underline{X} \times_{[G\backslash X]} \underline{X}$ is given by h, h', and $G \times S \to G \times S$, $(g,s) \mapsto (g\varphi(s), s)$ such that $\varphi(s)h'(s) = h(s)$. Again it is convenient to rewrite this as $h'(s) = \varphi(s)^{-1}h(s)$, so that $\underline{X} \times_{[G\backslash X]} \underline{X} \cong \underline{G} \times \underline{X}$, with this object going to (φ^{-1}, h). So the algebraic groupoid is $G \times X \rightrightarrows X$ where s is projection, t is action, $e = (e_G, 1_X)$, $m((\varphi^{-1},h),(\varphi'^{-1},h')) = (\varphi'^{-1}\varphi^{-1}, h)$, and $i(\varphi^{-1},h) = (\varphi, \varphi^{-1}\cdot h)$.

Example 3.10. Consider the CFG of elliptic curves $\mathfrak{X} = \mathcal{M}_{1,1}$ over $\mathcal{S} = (\mathrm{Sch}/\mathbb{C})$. Recall from §1.5 the modular families $C_\alpha \to S_\alpha$ of elliptic curves and the schemes $R_{\alpha,\beta}$ of pairs of points of S_α, S_β with isomorphism of the corresponding elliptic curves. The modular families are an object of $\mathcal{M}_{1,1}$ over $U := S_1 \amalg S_2$. We claim that $\underline{S_\alpha} \times_{\mathcal{M}_{1,1}} \underline{S_\beta} \cong \underline{R_{\alpha,\beta}}$. Then it follows that $\underline{U} \times_{\mathcal{M}_{1,1}} \underline{U} \cong \underline{R}$ where $R := \coprod_{1\le i,j\le 2} R_{i,j}$, and we recover the algebraic groupoid $R \rightrightarrows U$ that we had claimed would be an atlas for $\mathcal{M}_{1,1}$.

The morphism $\underline{S}_\alpha \to \mathcal{M}_{1,1}$ associates to $T \to S_\alpha$ the family of elliptic curves $T \times_{S_\alpha} C_\alpha \to T$. This comes with a section $\sigma_\alpha\colon T \to T \times_{S_\alpha} C_\alpha$. We need to exhibit a bijection between isomorphisms $\varphi\colon T \times_{S_\alpha} C_\alpha \to T \times_{S_j} C_j$ over T and morphisms $T \to R_{\alpha,j}$. For this, it suffices to treat the case that T is affine, $T = \mathrm{Spec}(A)$. We have an isomorphism $\varphi^*\mathcal{O}(3\sigma_\beta) \cong \mathcal{O}(3\sigma_\alpha)$. This is determined up to automorphism of $\mathcal{O}(3\sigma_\alpha)$, i.e., an element of $\mathcal{O}^*(T \times_{S_\alpha} C_\alpha) = A^*$. Expressing each family of curves by an equation in Weierstrass form corresponds to a particular kind of choice of basis for the space of global sections of $\mathcal{O}(3\sigma_\beta)$, resp. $\mathcal{O}(3\sigma_\alpha)$. With respect to these bases, pullback of global sections by φ corresponds to an element of $PGL_3(A)$. The Weierstrass equations constrain this to be of diagonal form, say with diagonal entries λ, μ, 1, and we are reduced to the computations of §1.5.

The first step in turning a groupoid scheme into a stack is to associate, in a simple way, a CFG to a given groupoid scheme $R \rightrightarrows U$. This CFG will be denoted $[R \rightrightarrows U]^{\mathrm{pre}}$. It won't quite be a stack, but it will be a *prestack*, a term that will be defined in §4.1 and that explains our choice of notation. A further step called *stackification* will be required to turn this prestack into a stack. If the groupoid scheme satisfies certain hypotheses, then this stack will be an algebraic stack. These steps will make up the other half of the dictionary between algebraic stacks and groupoid schemes.

Definition 3.11. Let $R \rightrightarrows U$ be a groupoid scheme. The **associated prestack** will be the CFG $[R \rightrightarrows U]^{\mathrm{pre}}$ defined as follows. An object over a scheme S is a morphism $g\colon S \to U$. A morphism over $f\colon S' \to S$ from $g'\colon S' \to U$ to $g\colon S \to U$ is a morphism $\gamma\colon S' \to R$ satisfying $s \circ \gamma = g'$ and $t \circ \gamma = g \circ f$. If $g\colon S \to U$ is an object of $[R \rightrightarrows U]^{\mathrm{pre}}$ then the identity morphism 1_g is given by $e \circ g\colon S \to R$. Composition of morphisms in $[R \rightrightarrows U]^{\mathrm{pre}}$ is defined using the multiplication map of the groupoid, as follows. If we have a pair of composable morphisms as pictured

$$\begin{tikzcd}
S'' \arrow[rrr, bend left, "\gamma'"] \arrow[drrr, "g''"] \arrow[dr, "f'"'] & & & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
& S' \arrow[urr, "\gamma"] \arrow[rr, "g'"'] \arrow[dr, "f"'] & & U \\
& & S \arrow[ur, "g"'] &
\end{tikzcd}$$

then $t \circ \gamma' = g' \circ f' = s \circ \gamma \circ f'$, so there is an induced morphism $(\gamma', \gamma \circ f')\colon S' \to R\; {}_t\!\times_s R$. Now we define $\gamma \circ \gamma' = m(\gamma', \gamma \circ f')$.

Exercise 3.2. (i) With this definition, $[R \rightrightarrows U]^{\mathrm{pre}}$ is a category, i.e., composition of morphisms is associative. (ii) This category, with the obvious functor to $\mathcal{S}$, is a CFG, with the pullback of an object determined just by composing morphisms of schemes.

Example 3.12. Let G be an algebraic group, and consider the classifying stack BG. We have already seen that $\underline{\Lambda} \times_{BG} \underline{\Lambda} \cong \underline{G}$, and the associated groupoid scheme is $G \rightrightarrows \Lambda$. Now $[G \rightrightarrows \Lambda]^{\mathrm{pre}}$ is equivalent to the category of *trivial* G-torsors. Indeed, there is just one object over any S in $\mathcal{S}$, and isomorphisms from this object to itself correspond bijectively with morphisms $S \to G$.

We see concretely why an extra step is necessary to recover the stack BG. What goes wrong with $[G \rightrightarrows \Lambda]^{\mathrm{pre}}$ is that all the nontrivial G-torsors are missing! Indeed, the definition of G-torsor includes the requirement to be locally trivial for the given topology (in the first examples, this will be the étale topology). So far, the topology has not entered into any of our constructions. The topology will play an essential role in the definition of a stack, and in the procedure for recovering a stack from an atlas presentation.

3. Constructions with groupoid schemes

Groupoid schemes provide a means of carrying out explicit constructions which mirror what takes place in the world of stacks. Because of their explicit nature, carrying out these constructions provides a good way of getting a sense of how stacks behave. We focus on three concrete constructions.

Example 3.13. There is an algebraic groupoid realization of fiber products. Let $R \rightrightarrows U$, $R' \rightrightarrows U'$, and $R'' \rightrightarrows U''$ be groupoid schemes, and let morphisms of groupoid schemes (ϕ', Φ') and (ϕ'', Φ'') to $R \rightrightarrows U$ from $R' \rightrightarrows U'$, resp. from $R'' \rightrightarrows U''$, be given. Then the category

$$[R' \rightrightarrows U']^{\mathrm{pre}} \times_{[R \rightrightarrows U]^{\mathrm{pre}}} [R'' \rightrightarrows U'']^{\mathrm{pre}} \tag{1}$$

is isomorphic to the category

$$[R' \times_U R \times_U R'' \rightrightarrows U' \times_U R \times_U U'']^{\mathrm{pre}}. \tag{2}$$

This is the groupoid where, in these fiber products the scheme to the left of U maps to U by a "target" map, and the scheme to the right maps by a "source" map (e.g., the first fiber product involves R' mapping to U by $t \circ \Phi'$) and where the source and target maps send (r', r, r'') to $(s'(r'), m(\Phi'(r'), r), s''(r''))$ and $(t'(r'), m(r, \Phi''(r'')), t''(r''))$, respectively. Indeed, an object over S of the fiber product (1) is a map $S \to U'$, a map $S \to U''$, and an isomorphism (map $S \to R$ whose composition with s is $S \to U' \to U$ and whose composition with t is $S \to U'' \to U$). That is precisely a morphism $S \to U' \times_U R \times_U U''$. A morphism over $\widetilde{S} \to S$ from $(\tilde{u}', \tilde{r}, \tilde{u}'')$ to (u', r, u'') will be a pair of morphisms, i.e., R'- and R''-valued points over T. Calling these r' and r'', the compatibility condition is the commutativity of the following square:

$$\begin{tikzcd}
\phi'(\tilde{u}') \arrow[r, "\Phi'(r')"] \arrow[d, "\tilde{r}"'] & \phi'(u') \arrow[d, "r"] \arrow[dl, dashed] \\
\phi''(\tilde{u}'') \arrow[r, "\Phi''(r'')"'] & \phi''(u'')
\end{tikzcd}$$

The middle factor R in $R' \times_U R \times_U R''$ corresponds to a choice of dotted arrow, so that $\tilde{r}$ and r are recovered as compositions of arrows — this accounts for the appearance of m in the formulas for the source and target maps of the groupoid scheme (2). We have an isomorphism on the level of objects and morphisms; from this the reader can work out the formulas for the identity, multiplication, and inverse in (2).

Exercise 3.3. Work this out explicitly in the case that the morphisms (ϕ', Φ') and (ϕ'', Φ'') are both the morphism $(e_G, 1_\Lambda)$ from $\Lambda \rightrightarrows \Lambda$ to $G \rightrightarrows \Lambda$. The result should be a groupoid presentation for $\underline{G}$ (in the sense of Example 3.8). Notice that the correct answer is *not* $\Lambda \times_G \Lambda \rightrightarrows \Lambda \times_\Lambda \Lambda$.

Example 3.14. Given $R' \rightrightarrows U'$ and $R' \rightrightarrows U$, a morphism $\gamma\colon U' \to R$ will associate to every object of $[R' \rightrightarrows U']^{\text{pre}}$ a *morphism* in $[R \rightrightarrows U]^{\text{pre}}$ (by composition with γ). We can give an explicit description of the category

$$\mathrm{HOM}([R' \rightrightarrows U']^{\text{pre}}, [R \rightrightarrows U]^{\text{pre}}). \tag{3}$$

The objects are morphisms (ϕ, Φ) of groupoid schemes. The morphisms from (ϕ, Φ) to $(\tilde{\phi}, \widetilde{\Phi})$ corespond bijectively with morphisms of schemes $\gamma\colon U' \to R$ satisfying $s \circ \gamma = \phi$, $t \circ \gamma = \tilde{\phi}$, and $m(\gamma \circ s, \widetilde{\Phi}) = m(\Phi, \gamma \circ t)$.

Every morphism of groupoid schemes will determine a morphism of stacks. The converse is not true, as we have seen. For instance, given a G-torsor on a scheme X, a corresponding morphism $\underline{X} \to BG$ will be (isomorphic to) one that comes from a morphism from $X \rightrightarrows X$ to $G \rightrightarrows \Lambda$ only when the given G-torsor is trivial. However, if we take $U \to X$ to be an étale cover which trivializes the given G-torsor, and $R = U \times_X U$ (Example 3.1), then $R \rightrightarrows U$ will be a different atlas for $\underline{X}$ (Example 3.8), and there will exist a morphism from $R \rightrightarrows U$ to $G \rightrightarrows \Lambda$ reproducing, up to isomorphism, the given morphism $\underline{X} \to BG$.

Example 3.14 will yield, as an application, a method for computing the set of 2-morphisms between a pair of morphisms of stacks. If f and g are morphisms of stacks $\mathfrak{X}' \to \mathfrak{X}$, represented concretely by morphisms of groupoids from $R' \rightrightarrows U'$ to $R \rightrightarrows U$, then any 2-morphism $f \Rightarrow g$ will come from a unique morphism in the HOM-category (3) of associated CFGs. Now we know that this HOM-category can be described entirely in terms of morphisms of schemes.

Example 3.15. Here is a general construction which produces a new groupoid scheme starting with a groupoid scheme $R \rightrightarrows U$ and a morphism $U' \to U$. This construction encapsulates refinements of atlases (e.g., when $U' \to U$ is an étale covering map), as well as the atlases of sub-CFGs (e.g., when U' is an open subscheme of U). First, we recall that a morphism of groupoid schemes $R' \rightrightarrows U'$ to $R \rightrightarrows U$ that satisfies Conditions 1.3(i)–(ii) corresponds, morally at least, to an isomorphism of stacks. Given $R \rightrightarrows U$ and an arbitrary morphism $U' \to U$, we choose the R' dictated by Condition 1.3(i). Then we obtain a groupoid scheme $R' \rightrightarrows U'$, where $R' = R \times_{U \times U} (U' \times U')$, the projection to $U' \times U'$ is (s', t'), and $m'((r, u_1', u_2'), (\tilde{r}, u_2', u_3')) = (m(r, \tilde{r}), u_1', u_3')$.

Exercise 3.4. In case $R \rightrightarrows U$ is the atlas for $\mathcal{M}_{1,1}$ (Example 3.10) and $U' = \mathrm{Spec}(\mathbb{C})$ is a point mapping to $u_0 \in U$, then R' will be $\mathrm{Aut}(E_0)$ where E_0 is the elliptic curve corresponding to the point u_0.

To continue the discussion of Example 3.15, suppose now that $U' \hookrightarrow U$ is the inclusion of a (locally closed) subscheme, and suppose further that

$$s^{-1}(U') = t^{-1}(U'). \tag{4}$$

Then R' will be equal to $s^{-1}(U')$. When $R \rightrightarrows U$ is an atlas for an algebraic stack $\mathfrak{X}$, there will be a dictionary between algebraic substacks of $\mathfrak{X}$ and subschemes U' of U satisfying (4). Substacks of an algebraic stack will play a role analogous to subschemes of a scheme. For instance, for $[X/G]$, these will be the $[Y/G]$ as Y ranges over the G-invariant subschemes of X.

Just as a complex algebraic variety is made up of a collection of points (satisfying the defining equations of the variety), a stack satisfying appropriate hypotheses (e.g., a reduced Deligne–Mumford stack, separated and of finite type over $\mathrm{Spec}(\mathbb{C})$) will have "points", each of which is a copy of BG for some finite group G. The moduli stacks of curves and complex orbifolds described the Introduction can be thought of in this way, and one can thus get a rough picture of what a stack "looks like".

Condition (4) is significant; it fails, e.g., if U' is a point of U (for general $R \rightrightarrows U$). The next exercise indicates how to "saturate" U' to a bigger subscheme which will satisfy (4). So, for instance, in Exercise 3.4, a point of U will determine a point-like closed substack, isomorphic to $B(\mathrm{Aut}(E_0))$. The next exercise will tell us how to produce the corresponding closed subscheme of U that satisfies (4).

EXERCISE 3.5. Let U_0' be a subscheme of U. Define $U' = t(s^{-1}(U_0'))$. This satisfies (4), at least as subsets of R. [Hint: both sides are equal to $\mathrm{pr}_2((s \circ \mathrm{pr}_1)^{-1}(U_0'))$.]

In good cases, the U' produced in Exercise 3.5 will make sense as a subscheme of U, and (4) will hold as an equality of subschemes of R. For instance, usually s and t will be étale, or smooth, or flat and locally of finite presentation. Any of these is enough to guarantee that s and t are *open* morphisms. Then, whenever U_0' is an open subscheme of U, the scheme U' produced in Exercise 3.5 will be a larger open subscheme of U. It will satisfy the equality (4): both $s^{-1}(U')$ and $t^{-1}(U')$ will be equal to the same open subscheme $R' \subset R$. There will be a corresponding open substack, having a groupoid presentation $R' \rightrightarrows U'$. It will further emerge that the groupoid scheme $R_0' := R \times_{U \times U} (U_0' \times U_0') \rightrightarrows U_0'$ that arises by applying the construction of Example 3.15 to the morphism $U_0' \to U$, is *another* groupoid presentation for this substack.

In the next chapter, when we have the stackification $[R \rightrightarrows U]$ of $[R \rightrightarrows U]^{\mathrm{pre}}$ at our disposal, we will give statements concerning these stacks $[R \rightrightarrows U]$ which are analogous to the statements appearing in Examples 3.13 through 3.15.

Answers to Exercises

3.1. (a) $i \circ e = m \circ (e \circ s \circ (i \circ e), i \circ e) = m(e \circ t \circ e, i \circ e) = m(e, i \circ e) = e$. (b) $m \circ (e, e) = m \circ (e \circ s \circ e, e) = m \circ (e \circ s, 1_R) \circ e = e$. (c) Copy a proof that $(f \cdot g)^{-1} = g^{-1} \cdot f^{-1}$ in a group, or better, in a groupoid, with arrows $f \colon x \to y$, $g \colon y \to z$. Start with the analogue of $(g \cdot f)^{-1} \cdot (g \cdot f) = 1_x$, which is the identity $m \circ (m, i \circ m) = e \circ s \circ m$. Corresponding to $((g \cdot f)^{-1} \cdot g) \cdot f = 1_x$, we have the identity $m \circ (\mathrm{pr}_1, m \circ (\mathrm{pr}_2, i \circ m)) = e \circ s \circ m$ [which is true since $m \circ (\mathrm{pr}_1, m \circ (\mathrm{pr}_2, i \circ m)) = m \circ (m \circ (\mathrm{pr}_1, \mathrm{pr}_2), i \circ m)) = m \circ (m, i \circ m)$]. Next write $(g \cdot f)^{-1} \cdot g = f^{-1}$, and prove the identity $m \circ (\mathrm{pr}_2, i \circ m) = i \circ \mathrm{pr}_1$. Finally, multiply on the right by g^{-1}. This gives $m \circ (i \circ \mathrm{pr}_2, i \circ \mathrm{pr}_1) = m \circ (i \circ \mathrm{pr}_2, m \circ (\mathrm{pr}_2, i \circ m)) = m \circ (m \circ (i \circ \mathrm{pr}_2, \mathrm{pr}_2), i \circ m)) = m \circ (e \circ t \circ \mathrm{pr}_2, i \circ m)) = m \circ (e \circ t \circ \mathrm{pr}_2, i \circ m)) = m \circ (e \circ s \circ i \circ m, i \circ m)) = i \circ m$, as required. (d) The left-hand square comes from the following diagram with fiber squares

$$\begin{tikzcd} R \,{}_t{\times}_s\, R \arrow[r, "(i\circ \mathrm{pr}_1, m)", "\sim"'] \arrow[d, "\mathrm{pr}_1"'] & R \,{}_t{\times}_s\, R \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & R \arrow[d, "s"] \\ R \arrow[r, "i", "\sim"'] & R \arrow[r, "t"] & R \end{tikzcd}$$

while a similar diagram takes care of the right-hand square.

3.2. Given $S''' \to S'' \to S' \to S$, a triple of composable morphisms gives rise to $(\gamma'', \gamma' \circ f'', \gamma \circ f' \circ f'') \to R \,{}_t{\times}_s\, R \,{}_t{\times}_s\, R \,{}_t{\times}_s\, R$. Applying $m \circ (1_R \times m) = m \circ (m \times 1_R)$ gives associativity of composition of morphisms in $[R \rightrightarrows U]^{\mathrm{pre}}$.

3.3. This is $\Lambda \times_\Lambda G \times_\Lambda \Lambda \rightrightarrows \Lambda \times_\Lambda G \times_\Lambda \Lambda$, which is $G \rightrightarrows G$.

3.4. We have $R' = (s,t)^{-1}(u_0, u_0)$, which is the automorphism group of E_0.

3.5. We have $s^{-1}(t(s^{-1}(U_0'))) = \mathrm{pr}_2(\mathrm{pr}_1^{-1}(s^{-1}(U_0'))) = \mathrm{pr}_2(m^{-1}(s^{-1}(U_0')))$. The last step, showing this equals $t^{-1}(t(s^{-1}(U_0')))$, uses the fact from Exercise 3.1(d) that $R\,{}_t{\times}_s R$ is, by (pr_2, m), isomorphic to the fiber product of $t\colon R \to U$ with itself.