CHAPTER 1

Introduction

Our aim in this chapter is to describe informally a variety of concrete examples that show why stacks are needed, and to illustrate some of the key ingredients of stacks. We start with a brief discussion of the two natures of a stack: as categories, and as atlases/groupoids. In practice it is usually easy to define the appropriate category, but it requires some work, requiring knowledge of the geometry involved, to construct an atlas. Then we look at examples, where these and other "stacky" features can be seen. Many of these examples should be familiar to the reader in some setting. Some of them were important in the early history of stacks, so reading about them will also give a glimpse of this history. Most of these examples will reappear later in the book, and most of the ideas seen here will be developed systematically later. Depending on a reader's background, statements made without proof can be accepted as facts to be used for motivation, or proofs can be worked out as exercises.

Making the notion of stack precise requires a fair amount of rather abstract language, including such mouthfuls as "categories fibered in groupoids". Starting in the next chapter we will develop this language slowly and carefully, with precise versions of most of these and many other examples. We hope that seeing several examples will help the reader digest what is to follow. However, we emphasize that nothing that is done here is logically necessary for reading the rest of the book.

1. Stacks as categories

Stacks are defined with respect to some fixed category $\mathcal{S}$, called the *base category*. For example, $\mathcal{S}$ can be the category (Sch) of schemes (or schemes over some fixed base), or $(\mathbb{C}_{\mathrm{an}})$ of complex analytic spaces, or (Diff) of differentiable manifolds, or (Top) of topological spaces, or even the category (Set) of sets. A *stack* over $\mathcal{S}$ will be a category $\mathfrak{X}$ together with a functor $\mathfrak{X} \to \mathcal{S}$, satisfying some properties — most of which will be left until later to discuss. These properties will depend, in part, on a "topology" on $\mathcal{S}$. A *morphism* from one $\mathfrak{X} \to \mathcal{S}$ to another $\mathfrak{Y} \to \mathcal{S}$ is defined to be a functor from $\mathfrak{X}$ to $\mathfrak{Y}$ that commutes with the projections to $\mathcal{S}$.

We start with some examples of this.

Example 1.1A. **Objects (Schemes).** An object X in $\mathcal{S}$ determines a category $\mathfrak{X}$, whose objects are pairs (S, f), where S is an object in $\mathcal{S}$ and $f\colon S \to X$ is a morphism. A morphism from (S', f') to (S, f) in $\mathfrak{X}$ is given by a morphism $g\colon S' \to S$ such that $f \circ g = f'$. The functor $\mathfrak{X} \to \mathcal{S}$ takes an object (S, f) to S, and takes a morphism from (S', f') to (S, f) to the underlying morphism from S' to S. It is a basic fact of Grothendieck/Yoneda that this category $\mathfrak{X}$ determines X up to canonical isomorphism.

This category will be denoted by $\underline{X}$; when the idea of stacks has been thoroughly digested, it can be denoted simply by X, but we will avoid doing this in Part I. We will be primarily interested in the case when X is a scheme and $\mathcal{S}$ is a category of schemes, but the notion is valid for any base category $\mathcal{S}$.

This example is a variation of Grothendieck's idea of replacing a scheme X by its functor of points. This is the contravariant functor h_X from $\mathcal{S}$ to (Set) with $h_X(S) = \mathrm{Hom}_{\mathcal{S}}(S, X)$, called the set of *$S$-valued points* of X. A morphism $q\colon X \to Y$ in $\mathcal{S}$ determines a natural transformation from h_X to h_Y, taking $f\colon S \to X$ to $q \circ f\colon S \to Y$. Just as X can be recovered from h_X, the q can be recovered from the natural transformation.

EXAMPLE 1.1B. **Torsors.** We start with $\mathcal{S} =$ (Top), the category of topological spaces, and let G be a topological group. A *G-torsor*, or *principal G-bundle*, is a (continuous) map $E \to S$, with a (continuous) action of G on E, which we take to be a right action; one requires that it be locally trivial, in the sense that S has an open covering $\{U_\alpha\}$ such that the restriction $E|_{U_\alpha}$ is isomorphic to the trivial bundle $U_\alpha \times G \to U_\alpha$. One has a category, which we denote by BG, whose objects are the G-torsors $E \to S$. (We will explain this notation later.) A morphism from $E' \to S'$ to $E \to S$ is given by a pair of maps $E' \to E$ and $S' \to S$ with the map $E' \to E$ being equivariant (commuting with the action of G) and the induced diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

being cartesian (this means that it commutes and the induced map from E' to the fibered product $S' \times_S E$ is a homeomorphism). The functor from BG to $\mathcal{S}$ is the obvious one that forgets the torsors, i.e., it takes an object $E \to S$ of BG to the object S of $\mathcal{S}$, and a morphism from $E' \to S'$ to $E \to S$ to the underlying map $S' \to S$.

There is an important generalization of this example. If G acts on the right on a space X, one defines a category, denoted $[X/G]$, whose objects are G-torsors $E \to S$, together with an equivariant map from E to X. A morphism from $E' \to S', E' \to X$ to $E \to S, E \to X$ is given by a pair of maps $E' \to E$ and $S' \to S$ giving a map of torsors as above, but, in addition, the composite $E' \to E \to X$ is required to be equal to the given map from E' to X. This may look rather arbitrary now, but we will soon see examples where these categories arise naturally. In this language, the category BG is the same as the category $[\bullet/G]$, where $\bullet$ is a point; and $[X/\{1\}]$ (where $\{1\}$ denotes the group with one element) is the same as $\underline{X}$. If G acts on the left on X, and we consider left G-torsors, we have similarly a category denoted $[G\backslash X]$.

This example (and its generalization) extend to the setting where G is a complex Lie group, and $\mathcal{S}$ is a category of smooth manifolds, or complex analytic spaces. In algebraic geometry, we can take $\mathcal{S}$ to be a category of schemes (all schemes, or schemes over a fixed base), and work with algebraic actions of algebraic groups. The major difference in the algebraic setting is that the notion of local triviality for a torsor is usually taken, not in the Zariski topology, but in the étale topology.

EXAMPLE 1.1C. **Moduli of curves.** Let $\mathcal{S}$ be the category of all schemes. A family of curves of genus g is a morphism $C \to S$ of schemes which is smooth and proper, whose geometric fibers are connected curves of genus g. The moduli stack $\mathcal{M}_g$ of curves of genus g has for its objects such families. A morphism from a family $C' \to S'$ to $C \to S$ is pair of morphisms $C' \to C$ and $S' \to S$ such that the induced diagram is cartesian, as in the case of torsors. The functor from $\mathcal{M}_g$ to $\mathcal{S}$ is the obvious one that forgets the families of curves.

In the case when $S' = S = \operatorname{Spec}(k)$, where k is a field, and $C' = C$, a curve over k, any automorphism of C over k will determine a morphism in $\mathcal{M}_g$ lying over the identity morphism of $\operatorname{Spec}(k)$. This illustrates the important point that the morphism from S' to S does *not* determine the morphism from C' gto C. Everything about the algebraic geometry of curves and their automorphisms is encoded in $\mathcal{M}_g \to \mathcal{S}$. It is precisely the existence of nontrivial automorphisms that prevents $\mathcal{M}_g$ from "being" a scheme.

There is a more classical object, the "coarse" moduli space M_g, which is a scheme (over $\operatorname{Spec}(\mathbb{Z})$); see [**75**]. Its geometric points correspond to isomorphism classes of curves, and it has the property that for any family of curves $C \to S$, there is a canonical morphism from S to M_g taking a geometric point s to the isomorphism class of the fiber C_s. These morphisms determine a functor from $\mathcal{M}_g$ to the category $\underline{M_g}$ determined by M_g. Moreover, M_g is characterized by a universal property that can be described as follows. Any morphism from $\mathcal{M}_g$ to the category $\underline{N}$ of a scheme N must factor uniquely through $\underline{M_g}$ (cf. [**75**], §5.2); note that the morphism from $\underline{M_g}$ to $\underline{N}$ is given by a morphism from M_g to N. In the language of stacks, the space M_g will be a "coarse moduli space" for the stack $\mathcal{M}_g$.

Many important examples of stacks will be variations of this example. For example, there is a stack $\mathcal{M}_{g,n}$ whose objects are families of curves $C \to S$ together with n pairwise disjoint sections $\sigma_1, \ldots, \sigma_n$ from S to C; the morphisms in $\mathcal{M}_{g,n}$ must be compatible with these sections. There are also "compactifications", which allow fibers to be nodal curves, with an appropriate notion of stability. One can also replace curves by other varieties.

The use of stacks in [**20**] to prove the irreducibility of the variety $M_g(k)$ of curves of genus g over any algebraically closed field k can be sketched as follows. Take $\mathcal{S}$ to be all schemes. Suppose for the moment that $\mathcal{M}_g$ were represented by a scheme M_g that is smooth over $\operatorname{Spec}(\mathbb{Z})$, and that $\mathcal{M}_g$ had a compactification $\overline{\mathcal{M}_g}$ (using stable curves) that is represented by a scheme $\overline{M}_g$ that contains M_g as an open subscheme, with $\overline{M}_g$ smooth and projective over $\operatorname{Spec}(\mathbb{Z})$. The classical fact that $\overline{M}_g(\mathbb{C})$ is connected would imply, by a connectedness theorem of Enriques and Zariski, that all geometric fibers $\overline{M}_g(k)$ of $\overline{M}_g$ over $\operatorname{Spec}(\mathbb{Z})$ are connected. Since a nonsingular connected variety is irreducible, the open subvariety $M_g(k)$ would also be irreducible. Although these assertions are all false for the coarse moduli spaces — even $M_g(\mathbb{C})$ is singular — they are true, suitably interpreted, for the corresponding stacks, and the irreducibility of the coarse varieties $M_g(k)$ follows.

The notion of S-valued points, discussed in Example 1.1A, can be used to make casual set-theoretic notation rigorous. For example, if $\mathcal{S}$ is the category schemes over a base Λ, and a group scheme G over Λ acts on the right on a scheme X in $\mathcal{S}$, the associativity condition "$(x \cdot g) \cdot h = x \cdot (g \cdot h)$" is not strong enough if applied pointwise, but it is if applied to S-valued points for all S in $\mathcal{S}$. Here x, g, and h are taken to be in $h_X(S)$, $h_G(S)$, and $h_G(S)$, and $x \cdot g$ denotes the composite $S \xrightarrow{(x,g)} X \times_\Lambda G \xrightarrow{\sigma} X$, where σ is the action. The equation "$(x \cdot g) \cdot h = x \cdot (g \cdot h)$" for all such x, g, and h is equivalent to the commutativity of the diagram

$$\begin{tikzcd} X \times_\Lambda G \times_\Lambda G \arrow[r, "\sigma \times 1_G"] \arrow[d, "1_X \times m"'] & X \times_\Lambda G \arrow[d, "\sigma"] \\ X \times_\Lambda G \arrow[r, "\sigma"] & X \end{tikzcd}$$

where $m\colon G \times_\Lambda G \to G$ is the product in G; one sees this by considering the universal case where S is $X \times_\Lambda G \times_\Lambda G$, with x, g, and h the three projections. We will often use such abbreviations in this text.

The idea of stacks as categories leads to some complications. Morphisms should then be functors, specifically, morphism from one stack to another will be functors that commute with the given functors to the base category. But in the world of categories, the natural notion of isomorphism is not a strict isomorphism (bijection on the level of objects and morphisms), but rather an *equivalence* of categories. So, quite different looking categories can give rise to isomorphic stacks. While stacks are important tools in algebraic geometry, it is not easy to do algebraic geometry on a category! For example, we would like to say that $\mathcal{M}_g$ is smooth, and that it is an open substack of a smooth compactification, whose complement is a divisor with normal crossing. And we would like to describe line bundles and vector bundles on these stacks, and do intersection theory on them. Only after a considerable amount of preparation will we see how to do these things.

2. Stacks from groupoids

An algebraic stack will come from a kind of atlas, which is called a groupoid. If $\mathcal{S}$ is the base category, a *groupoid in* $\mathcal{S}$, or an $\mathcal{S}$*-groupoid*, consists of a pair of objects U and R in $\mathcal{S}$, together with five morphisms: s (the "source") and t (the "target") from R to U, e (the "identity") from U to R, a morphism m (the "multiplication") from the fibered product[1] $R\ {}_t{\times}_{U,s} R$ to R, and a morphism i (the "inverse") from R to R, which satisfy some natural axioms.

In fact, you already know how to write down these axioms, as follows. Take a category in which all morphisms are isomorphisms and let U be the objects of this category, and R the morphisms or arrows, with s and t be the usual source and target (sending $f \in R$ to its source and target respectively), e the identity (taking an object to the identity map on it), m the composition (taking a pair $f \times g$ to $g \circ f$), and i

[1] For this to make sense, we assume, at least for now, that the fibered product of R with itself over U, using the two projections t and s, must exist in the category $\mathcal{S}$; we usually abbreviate this to $R\ {}_t{\times}_s R$. Similarly whenever we write cartesian products such as $U \times U$, we are assuming they exist as well.

the inverse. The axioms for a category amount to certain compatibilities among these morphisms, such as $s \circ e = \mathrm{id}_U$. If you write down these compatibilities then you get exactly the axioms for a groupoid.

EXERCISE 1.1. Do this now and try to obtain axioms for a groupoid. (You can check against the list at the beginning of Chapter 3 to see if you have missed any.)

A notation like (U, R, s, t, e, m, i) for a groupoid is too unwieldy to be practical. We will often use the notation $R \rightrightarrows U$ for a groupoid with spaces U and R indicated, with the two arrows (for s and t) standing as an abbreviation for all five maps. In fact, e and i are uniquely determined by s, t, and m, so we often leave their construction to the reader. When $\mathcal{S} = (\mathrm{Top})$, we call this a *topological groupoid*, and when $\mathcal{S} = (\mathbb{C}_{\mathrm{an}})$, we call it an *analytic groupoid*. When $\mathcal{S}$ is a category of schemes, it will be called an *algebraic groupoid* or a *groupoid scheme*.

The *isotropy group* $\mathrm{Aut}(x)$ of a point x in U is the set $s^{-1}(x) \cap t^{-1}(x) \subset R$, which is a group with product determined by m.

A morphism from a groupoid $R' \rightrightarrows U'$ to a groupoid $R \rightrightarrows U$ is given by a pair (ϕ, Φ) of morphisms $\phi\colon U' \to U$ and $\Phi\colon R' \to R$ commuting with all the morphisms of the groupoid structure.

One geometric example of a groupoid, called the *fundamental groupoid* of a topological space, is probably familiar to you. Although it will not play much of a role in this book, it shows clearly the not-everywhere-defined grouplike structure of a groupoid. If X is a topological space, its fundamental groupoid can be denoted $\Pi(X) \rightrightarrows X$. The elements of $\Pi(X)$ are triples (x, y, σ), with x and y points of X and σ a homotopy class of paths in X starting at x and ending at y; s and t take this triple to x and y respectively, and $m((x, y, \sigma), (y, z, \tau)) = (x, z, \sigma * \tau)$, where $\sigma * \tau$ is the usual product coming from tracing first a path representing σ and then a path representing τ.[2] This groupoid has advantages over the usual fundamental group (which requires an arbitrary choice of base point), particularly in the study of the Van Kampen theorem when the intersection of the open sets involved is not connected (cf. [**16**]). There are also useful variants of the fundamental groupoid, such as the groupoid $\Pi(X, A) \rightrightarrows A$, where A is a subset of X, and the paths connect points of A. If X is a foliated manifold, one can require the paths and homotopy equivalences to lie within leaves of the foliation; if one replaces homotopy equivalence by holonomy equivalence, one arrives at the *holonomy groupoid* of the foliation [**43**].

A continuous mapping $f\colon X \to Y$ determines a morphism (f, F) from the groupoid $\Pi(X) \rightrightarrows X$ to the groupoid $\Pi(Y) \rightrightarrows Y$, with $F(\sigma) = f \circ \sigma$. Then a homotopy $H\colon X \times [0, 1] \to Y$ from f to g determines a mapping $\theta\colon X \to \Pi(Y)$, taking x in X to the path $t \mapsto H(x, t)$. If likewise (g, G) denotes the morphism of groupoids determined by g, this mapping θ satisfies the identities

$$s(\theta(x)) = f(x), \quad t(\theta(x)) = g(x), \quad \text{and} \quad \theta(s(\sigma)) \cdot G(\sigma) = F(\sigma) \cdot \theta(t(\sigma))$$

[2]For a general space, its fundamental groupoid is a groupoid of sets. If X has a universal covering space, i.e., X is semilocally simply connected, then $\Pi(X)$ has a natural topology so that s and t are local homeomorphisms, and the fundamental groupoid is a topological groupoid.

for x in X and σ in $\Pi(X)$. Maps θ satisfying these identities are called 2-*isomorphisms*; they will be the analogues of homotopies for groupoids.

Example 1.2A. **Classical atlases.** If X is a scheme, or manifold, or topological space, and $\{U_\alpha\}$ is an open covering of X (with α varying in some index set), let $U = \coprod U_\alpha$ be the disjoint union, and let $R = \coprod U_\alpha \cap U_\beta$, the disjoint union of all intersections over all ordered pairs (α, β); equivalently, $R = U \times_X U$. The five maps are the obvious ones: s takes a point in $U_\alpha \cap U_\beta$ to the same point in U_α, and t takes it to the same point in U_β; e takes a point in U_α to the same point in $U_\alpha \cap U_\alpha$; i takes a point in $U_\alpha \cap U_\beta$ to the same point in $U_\beta \cap U_\alpha$; for m, if u is in $U_\alpha \cap U_\beta$ and v is in $U_\delta \cap U_\gamma$, requiring $t(u)$ to equal $s(v)$ says that $\beta = \delta$ and $u = v$, so we can set $m(u, v) = u = v$ in $U_\alpha \cap U_\gamma$.

The basic construction of algebraic geometry of *recollement* (gluing) amounts to constructing X from a compatible collection of schemes $\{U_\alpha\}$, with isomorphisms from an open set $U_{\alpha\beta}$ of each U_α to an open set $U_{\beta\alpha}$ of U_β, satisfying axioms of compatibility. These axioms are the same as those for constructing a manifold by gluing open subsets of Euclidean spaces.

There is a similar atlas (groupoid) constructed from an étale covering $\{U_\alpha \to X\}$, but taking R to be $\coprod U_\alpha \times_X U_\beta$. In fact, for any morphism $U \to X$, one can construct a groupoid, with $R = U \times_X U$, with s and t the two projections, e the diagonal, i the map reversing the two factors, and m the composite

$$(U \times_X U) \times_U (U \times_X U) \cong U \times_X U \times_X U \to U \times_X U,$$

where the second map is the projection $p_{1,3}$ to the outside factors. Applying this to the case of an open covering $U = \coprod U_\alpha \to X$ recovers the "gluing" atlas.

A trivial but important special case of this construction takes, for any object X of our category $\mathcal{S}$, the groupoid arising from the identity map from X to X. Here $U = X$, $R = X$, and all the maps of the groupoid are identity maps. When $\mathcal{S}$ is the category of sets, so a groupoid is identified with a category, a set is exactly a category in which the only maps are identity maps. In this sense, one may say that schemes (or spaces) are to stacks as sets are to (groupoid) categories.

In this collection of examples, the canonical map $(s, t)\colon R \to U \times U$ is an embedding (a monomorphism), so that R defines an equivalence relation on U, and X may be thought of as the quotient of U by this equivalent relation. In fact, *algebraic spaces* are constructed from equivalence relations $R \to U \times U$ with projections s and t étale. (Any equivalence relation on a set U, in fact, determines a groupoid of sets.) One major difference between a scheme or algebraic space and a general stack is that, for an atlas for a stack, the morphism from R to $U \times U$ need not be one-to-one (on geometric points).

Example 1.2B. **Group actions.** Suppose an algebraic (resp. topological) group G acts on a scheme (resp. topological space) U, say on the right. There is a natural equivalence relation on U: two points u and v are equivalent if they are in the same orbit: $v = u \cdot g$ for some $g \in G$. There is a better groupoid to construct from this action: take $R = U \times G$, and think of a point (u, g) in R as being a point u together with an arrow g from u to $u \cdot g$. This indicates that we look at the atlas

$$U \times G \rightrightarrows U$$

where $s\colon U \times G \to U$ is the first projection and $t\colon U \times G \to U$ is the action (so $s(u,g) = u$ and $t(u,g) = u \cdot g$). For the remaining maps, e is the identity $(e(u) = (u, e_G))$,

$$m\big((u,g),(u \cdot g, h)\big) = (u, g \cdot h),$$

and $i(u,g) = (u \cdot g, g^{-1})$.

This groupoid is sometimes denoted by a semi-direct product notation $U \rtimes G$, and it is called a *transformation groupoid*. This groupoid will, in fact, be an atlas for the stack $[U/G]$ discussed in Example 1.1B. Note that for x in U, the isotropy group $\mathrm{Aut}(x)$ of the groupoid is the same as the isotropy or stabilizer group G_x for the group action. Whenever there are fixed points, the mapping $(s,t)\colon R \to U \times U$ is not an embedding: if $u \in U$ and $g \in G$, with $g \neq e_G$ and $u \cdot g = u$, then (u,g) and (u, e_G) have the same image. The stack determined by this groupoid will capture the action better than the naive quotient U/G, when this latter quotient exists. An extreme example is the action of G on a point $\bullet$; the groupoid $G \rightrightarrows \bullet$ carries the information of the group G (and the stack BG from Example 1.1B), but the quotient space is just the point $\bullet$.

An analogous groupoid $G \times U \rightrightarrows U$ arises from a left action of a group G on U. This groupoid, also denoted $G \ltimes U$, is defined by setting $s(g,u) = u$, $t(g,u) = g \cdot u$, and $m((g,u),(g', g \cdot u)) = (g' \cdot g, u)$. More generally, if G acts on the left on U, and H acts on the right on U, and the actions commute in the sense that $(g \cdot u) \cdot h = g \cdot (u \cdot h)$ for all $g \in G$, $u \in U$, and $h \in H$, there is a groupoid

$$G \times U \times H \rightrightarrows U,$$

with $s(g,u,h) = u$, $t(g,u,h) = g \cdot u \cdot h$, and $m((g,u,h),(g', g \cdot u \cdot h, h') = (g' \cdot g, u, h \cdot h')$. This groupoid may be denoted $G \ltimes U \rtimes H$.

Example 1.2C. **Curves in projective space.** Fix an integer $g \geq 2$. An important fact about moduli of curves is that curves of genus g can be uniformly embedded in projective space. This is based on the canonical sheaf (which, for a curve, is just the sheaf of differentials), an ample sheaf whose third tensor power is very ample. From the classical Riemann–Roch formula, it is computed that this gives an embedding of the curve into $\mathbb{P}^{5g-6}$. For any family of genus g curves $C \to S$, the sheaf $\omega_{C/S}^{\otimes 3}$ gives rise to an embedding of C in a projective bundle over S.

Inside the Hilbert scheme of $\mathbb{P}^{5g-6}$ there is a locus $\mathrm{Hilb}_{g,3}$, smooth of dimension $25g^2 - 47g + 21$, of tricanonically embedded curves of genus g. The canonical sheaf is preserved by automorphisms of curves, and all isomorphisms are given by projective linear transformations. The action of the projective linear group makes

$$PGL_{5g-5} \times \mathrm{Hilb}_{g,3} \rightrightarrows \mathrm{Hilb}_{g,3}$$

an atlas for $\mathcal{M}_g$. More classically, the moduli space M_g (a variety of dimension $3g-3$) is a quotient variety for this action of PGL_{5g-5}.

The proof of irreducibility of the moduli spaces $M_g(k)$ using stacks [**20**] makes use of the existence of another atlas $R \rightrightarrows U$ for $\mathcal{M}_g$, such that U and R are both smooth and have the same dimension $3g-3$ as M_g. Such an atlas exists; in fact, U can be taken to be the disjoint union of finitely many locally closed subvarieties of $\mathrm{Hilb}_{g,3}$ and R a corresponding disjoint union of subvarieties of $PGL_{5g-5} \times \mathrm{Hilb}_{g,3}$. The existence of such an atlas is an important, nontrivial fact which is linked to properties (coming from deformation theory) of curves of genus g.

For a group action on a variety, there might exist a classical quotient variety. But for some purposes the groupoid $U \rtimes G$ is better. In Example 1.2C, we saw that the groupoid has nice properties (e.g., smoothness) which do not hold for the quotient variety.

Let us compare the stack quotient with a more classical quotient. Here for simplicity the base category is taken to be $(\mathbb{C}_{\mathrm{an}})$. If a complex Lie group G acts on a complex space X, a *categorical quotient* is a complex space X/G, with a G-invariant surjective morphism $q\colon X \to X/G$ that satisfies a universal property: for any complex analytic space Y and any G-invariant morphism $f\colon X \to Y$, there is a unique morphism $\bar{f}\colon X/G \to Y$ such that $f = \bar{f} \circ q$.

We compare the quotients in the following example. Let $G = \mathbb{C}^\times$ act by $(x,y)\cdot t = (xt, yt)$ on $\mathbb{C}^2$, and also on $U := \mathbb{C}^2 \smallsetminus \{(0,0)\}$. Then:

(1) The map from U to $\mathbb{P}^1$ that sends (x,y) to $[x:y]$ identifies $\mathbb{P}^1$ as the categorical quotient U/G.

(1′) For any analytic space S, morphisms $S \to \mathbb{P}^1$ are in bijective correspondence with G-torsors over S equipped with a G-equivariant morphism to U, up to G-equivariant isomorphism commuting with the morphisms to U.

(2) The categorical quotient $\mathbb{C}^2/G$ is a point.

(2′) An analytic space S admits infinitely many G-torsors with equivariant maps to $\mathbb{C}^2$, up to G-equivariant isomorphism commuting with the maps to $\mathbb{C}^2$, while possessing always a unique map to a point.

(We just state these as facts for now; the techniques to give complete justifications will come later. Precisely analogous assertions hold as well in the topological and algebraic settings.) On U, where G acts freely, the classical quotient U/G represents the stack quotient $[U/G]$. In the world of stacks, $[\mathbb{A}^2/G]$ will contain $[U/G]$ as a dense open substack, as contrasted with the classical notion of categorical quotient, in which $\mathbb{A}^2/G$ is a point.

As one would expect from the case of manifolds, many different groupoids can be atlases for the same stack. Example 1.2C made reference to two different groupoids for $\mathcal{M}_g$: there were maps $U \to \mathrm{Hilb}_{g,3}$ and $R \to PGL_{5g-5} \times \mathrm{Hilb}_{g,3}$ (componentwise inclusion maps), giving rise to a map of groupoids from $R \rightrightarrows U$ to $PGL_{5g-5} \times \mathrm{Hilb}_{g,3} \rightrightarrows \mathrm{Hilb}_{g,3}$. Of course, an arbitrary map of groupoids (ϕ, Φ) from $R' \rightrightarrows U'$ to $R \to U$ will not determine an isomorphism of their corresponding stacks. There are two properties that will guarantee this, the first corresponding to injectivity, the second to surjectivity. The properties are:

Condition 1.3(i). The diagram

$$\begin{tikzcd} R' \arrow[r, "{(s,t)}"] \arrow[d, "\Phi"'] & U' \times U' \arrow[d, "\phi\times\phi"] \\ R \arrow[r, "{(s,t)}"] & U \times U \end{tikzcd}$$

must be cartesian.

Condition 1.3(ii). For every $u \in U$ there is a $u' \in U'$ and an $a \in R$ such that $s(a) = \phi(u')$ and $t(a) = u$; or in other words, the morphism

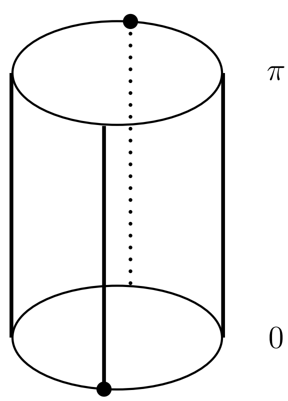
$$V \ = \ U' \,{}_{\phi}\!\times_{U,s} R \ \longrightarrow \ U$$

determined by t must be surjective.

Condition (i) can be expressed in terms of S-valued points: the map $h_{R'}(S) \to h_R(S) \times_{h_{U\times U}(S)} h_{U'\times U'}(S)$ is a bijection for all S. In condition (ii), "surjective" must be interpreted correctly. Requiring surjectivity on the naive point level is too weak, and requiring that $h_V(S) \to h_U(S)$ be surjective for all S is too strong, since that is equivalent to the existence of a splitting morphism from U to V. (For example, a fiber bundle projection should be surjective, but it may have no global section.) What works is to require that the map must be locally surjective, using the topology on $\mathcal{S}$. That is, we require U to have a covering $\{U_\alpha \to U\}$ such that each $U_\alpha \to U$ factors through V.

In the case where $\mathcal{S}$ is the category of sets (with the discrete topology), so the groupoids are categories and maps between them are functors, condition (i) says that this functor is fully faithful, and condition (ii) says that it is essentially surjective; together they say that the functor is an equivalence of categories.

Example 1.4. We conclude this discussion with a geometric example. Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$, and let X be the cylinder $D \times \mathbb{R}$, with the identification $(z, \phi) \sim (z', \phi')$ if $\phi' - \phi = n\pi$, $n \in \mathbb{Z}$, and $z' = (-1)^n z$.



The group S^1 acts on X, by $e^{i\vartheta} \cdot (z, \varphi) = (z, \vartheta + \varphi)$. (This is an example of a "Seifert circle bundle".) The group $\{\pm 1\}$ acts on D by $(-1) \cdot z = -z$, and $\{\pm 1\}$ is a subgroup of S^1 by $-1 \mapsto e^{i\pi}$. The embedding $D \to X$, $z \mapsto (z, 0)$, is equivariant with respect to $\{\pm 1\} \to S^1$, giving a morphism of groupoids

$$\{\pm 1\} \ltimes D \;\to\; S^1 \ltimes X.$$

Exercise 1.2. (a) Show that this morphism satisfies properties (i) and (ii), where the base category $\mathcal{S}$ is (Top) or (Diff). (b) Compute the isotropy groups of these actions at all points.

3. Triangles

Mike Artin has suggested that a quick way to get a feeling for stacks is to work out what the moduli space of ordinary triangles should be. As in all moduli problems, it is important to consider families of objects, in this case plane triangles up to isometry. If S is a topological space, a family of triangles over S will be a continuous and proper map $X \to S$, making X a fiber bundle over S with a continuously varying metric on fibers[3], such that each fiber is (isometric to) a triangle.

The classical moduli space of triangles would simply be the set T of plane triangles, up to isometry, suitably topologized. As a set, T consists of triples (a, b, c) of side lengths, satisfying (strictly) the triangle inequalities, up to reordering. As a space, T is a quotient of a subset of Euclidean space. Indeed, consider the open cone

$$\widetilde{T} = \{\, (a, b, c) \in \mathbb{R}^3_+ \mid a + b > c,\; b + c > a,\; c + a > b \,\}.$$

Then we have a map $\widetilde{T} \to T$, and T inherits a topology from $\widetilde{T}$, the quotient topology.

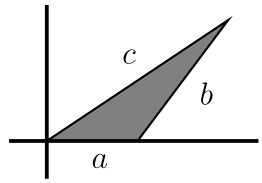
To phrase this moduli problem in the categorical language, we take $\mathcal{S}$ to be the category of topological spaces, and define a category $\mathfrak{T}$ whose objects are families of triangles $X \to S$. A morphism in $\mathfrak{T}$ from one family $X' \to S'$ to another family $X \to S$ is given by a pair of (continuous) maps $X' \to X$ and $S' \to S$ such that the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

commutes, and so that the induced maps on the fibers are isometries. The functor from $\mathfrak{T}$ to $\mathcal{S}$ is the evident one, as in the examples of Section 1.1.

The moduli problem becomes easier if we consider, instead, *ordered* triangles, ordering the sides (or, equivalently, their opposite vertices). Here the objects of the corresponding category $\widetilde{\mathfrak{T}}$ would be fibrations $X \to S$ as before, together with a triple (α, β, γ) of sections that pick out the three vertices of each fiber; the morphisms are required to be compatible with these sections. The moduli space is then the open cone $\widetilde{T} \subset \mathbb{R}^3$. There is a universal family $\widetilde{Y} \subset \widetilde{T} \times \mathbb{R}^2$, with its projection $\widetilde{Y} \to \widetilde{T}$, and with the fiber over (a, b, c) in $\widetilde{T}$ being the triangle

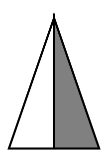
[3] That is, a continuous distance function $d\colon X \times_S X \to \mathbb{R}_{\geq 0}$ whose restriction to $X_s \times X_s$ is a metric on the fiber X_s, for every $s \in S$.



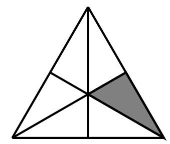
The essential point is that any triangle with labeled edges of lengths a, b, and c is *canonically* isometric to this one: given any family $X \to S$ with three vertex sections, there is a unique map from S to $\widetilde{T}$, and a unique isomorphism of X with the pullback of this universal family. Having this universal family means, in classical language, that $\widetilde{T}$ is a *fine moduli space*. In the language of stacks, there is an isomorphism of stacks (equivalence of categories) between $\widetilde{\mathfrak{T}}$ and $\underline{\widetilde{T}}$.

If we want a moduli space for unordered triangles, however, the situation is more complicated. The symmetric group $\mathfrak{S}_3$ acts (on the right) on $\widetilde{T}$ by permuting the coordinates, and the quotient space $T = \widetilde{T}/\mathfrak{S}_3$ is the obvious candidate for a moduli space of triangles. Its points, at least, do correspond to triangles up to isometry. The group $\mathfrak{S}_3$ also acts on $\widetilde{Y}$, compatibly with its projection to $\widetilde{T}$. We can therefore construct $Y = \widetilde{Y}/\mathfrak{S}_3$, with an induced map $Y \to T$. If one were trying to construct a universal family of plane triangles, this would be a first guess.

Any family of triangles $X \to S$ will determine a map from S to T, but the family may not be isomorphic (uniquely, or even at all) to the pullback of $Y \to T$. In classical language, then, this moduli space T is a *coarse*, but not a fine, moduli space, for $\mathfrak{T}$. For example, when S is the circle S^1 and $X \to S$ is a family of equilateral triangles that rotates the triangle by 120° in one revolution around the circle, then this is not a constant family even though the corresponding map from S to T is constant. For an isosceles triangle (taking S to be a point), say with sides of lengths 1, 2, and 2, corresponding to a point t in T, there are three points $(1, 2, 2)$, $(2, 1, 2)$, and $(2, 2, 1)$ in $\widetilde{T}$ lying over t; the action of the group includes flips over the altitude, and the fiber of Y over t is the quotient of the triangle by this flip:



For an equilateral triangle, there is only one point of $\widetilde{T}$ over the point t in T, and the fiber of Y over t is the quotient of the triangle by the action of $\mathfrak{S}_3$:



In fact, $Y \to T$ fails to satisfy the definition of family of triangles (e.g., Y is not a fiber bundle over T). The problems with $Y \to T$ arise from triangles with nontrivial automorphisms.

EXERCISE 1.3. Let $Y^\circ \to T^\circ$ be the restriction of $Y \to T$ to the locus T° of triangles with sides of distinct lengths. Show that $Y^\circ \to T^\circ$ is a fiber bundle and gives a universal family: T° is a fine moduli space for such triangles.

Given any family $X \to S$ of (unordered) triangles, let $\widetilde{S}$ be the space of pairs

$$(s, \text{ordering of the edges of } X_s).$$

Then $\widetilde{S} \to S$ is a 6-sheeted covering space, in fact, a principal bundle (torsor) under the symmetric group $\mathfrak{S}_3$. If $\widetilde{X} \to \widetilde{S}$ is the pullback of the given family $X \to S$ by the covering map $\widetilde{S} \to S$, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & \widetilde{T} \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array}$$

where the map $\widetilde{S} \to \widetilde{T}$ commutes with the action of $\mathfrak{S}_3$. This is exactly the data for an object of the stack $[\widetilde{T}/\mathfrak{S}_3]$ described in Section 1.1: the stack $\mathfrak{T}$ is isomorphic to the quotient stack $[\widetilde{T}/\mathfrak{S}_3]$. (The reader may verify that the functor from $\mathfrak{T}$ to $[\widetilde{T}/\mathfrak{S}_3]$ is an equivalence of categories.) The transformation groupoid $\widetilde{T} \times \mathfrak{S}_3 \rightrightarrows \widetilde{T}$ will be an atlas for this stack.

As in this example, it frequently happens that a coarse moduli space can be constructed as a quotient U/G of a space U by the action of a group G. This crude quotient space cannot capture the geometry of the moduli problem near points u of U where the stabilizer $G_u = \{g \in G |\ g \cdot u = u\}$ is not trivial. The stack is designed to remember some part of the group action. The group action is not part of the information carried by the stack, however. Indeed, if it were, we would just be studying equivariant spaces.

Here is quite a different atlas for the same stack. By a *plane triangle* we mean a triangle embedded in $\mathbb{R}^2$. Let G be the Lie group of isometries of $\mathbb{R}^2$, which is the 3-dimensional group generated by rotations, reflections, and translations. Let V be the space of (unordered) plane triangles, which is a 6-dimensional manifold.[4] We have a universal family $Z \subset \mathbb{R}^2 \times V$ of plane triangles over V. Note that G acts on the left on V, and on $\mathbb{R}^2 \times V$, preserving Z.

[4]This can be constructed as a quotient of the set $\widetilde{V}$ of noncollinear triples in $(\mathbb{R}^2)^3$ by the action of $\mathfrak{S}_3$. That V is a manifold follows from the general fact that this action is free.

We claim that the stack $\mathfrak{T}$ is isomorphic to the quotient stack $[G\backslash V]$. Indeed, if $X \to S$ is an object of $\mathfrak{T}$, there is a principal (left) G-bundle $E \to S$, whose fiber over s is the space of all isometric embeddings of the fiber X_s into $\mathbb{R}^2$. (Note that this G-torsor is trivial over any open set of S on which the $\mathfrak{S}_3$-covering $\widetilde{S} \to S$ is trivial.) We have a G-equivariant map from E to V, since any point of E determines a plane triangle. This gives a functor from $\mathfrak{T}$ to $[G\backslash V]$, which is an equivalence of categories. Summarizing, we have isomorphisms of stacks:

$$[G\backslash V] \cong \mathfrak{T} \cong [\widetilde{T}/\mathfrak{S}_3].$$

Note that the two corresponding atlases even have different dimensions. However, $\dim V - \dim G = 6 - 3$ and $\dim \widetilde{T} - \dim \mathfrak{S}_3 = 3 - 0$ are equal; this stack $\mathfrak{T}$ will be 3-dimensional.

We can also see a direct relation between the groupoid $G \times V \rightrightarrows V$ and the category $\mathfrak{T}$. Any family of triangles is locally planar: if $X \to S$ is a family of triangles, we can choose an open covering $\{U_\alpha\}$ of S, with maps $\phi_\alpha\colon U_\alpha \to V$ and an isomorphism of $X|_{U_\alpha}$ with the pullback of $Z \to V$. Compatible with these, there are, on $U_\alpha \cap U_\beta$, unique maps $\Phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ such that $\phi_\beta(p) = \Phi_{\beta\alpha}(p) \cdot \phi_\alpha(p)$. This gives a map $\Phi\colon \coprod U_\alpha \cap U_\beta \to G \times V$, taking p in $U_\alpha \cap U_\beta$ to $(\Phi_{\beta\alpha}(p), \phi_\alpha(p))$.

Exercise 1.4. Show that $\{\phi_\alpha\}$ and $\{\Phi_{\alpha\beta}\}$ determine a morphism from the groupoid $\coprod U_\alpha \cap U_\beta \rightrightarrows \coprod U_\alpha$ to the groupoid $G \times V \rightrightarrows V$. The first is an atlas for S as in Example 1.2A, the second an atlas for $[G\backslash V]$.

The following exercise shows how the two atlases for $\mathfrak{T}$ are related.

Exercise 1.5. The set $\widetilde{V}$ of noncollinear triples in $(\mathbb{R}^2)^3$ has a right action of $\mathfrak{S}_3$ compatible with the left action of G. Construct morphisms of groupoids from the groupoid $G \times \widetilde{V} \times \mathfrak{S}_3 \rightrightarrows \widetilde{V}$ to the groupoid $G \times V \rightrightarrows V$ and to the groupoid $\widetilde{T} \times \mathfrak{S}_3 \rightrightarrows \widetilde{T}$, and show that they satisfy Conditions 1.3(i)–(ii).

Exercise 1.6. How do the results of this section change if one replaces isometry (congruence) of triangles by similarity?

4. Conics

We want to classify *conics*; for us a conic will be a curve which is isomorphic to the curve defined by a homogeneous polynomial of degree two in $\mathbb{P}^2$. Here we take $\mathcal{S}$ to be schemes over $\mathbb{C}$.

There are just three isomorphism classes of plane conics. Let x, y, z be the homogeneous coordinates on $\mathbb{P}^2$, and identify a plane conic with the homogeneous polynomial that defines it (identifying two polynomials if one is a nonzero multiple of the other). The isomorphism classes are

(1) N: nonsingular conics, e.g. $x^2 + y^2 + z^2$,
(2) L: pairs of two different lines, e.g. xy,
(3) D: double lines, e.g. x^2.

Therefore, in some sense the moduli space of plane conics is just a set $\{N, L, D\}$ of three points.

If M were a fine moduli space for conics, then the morphisms from a scheme S to M would be in one-to-one correspondence with the families of conics over S. If M were even a coarse moduli space, any family of conics over S would determine a morphism from S to M.

We can first see that if $\{N, L, D\}$ is such a moduli space, then it cannot carry the discrete topology. In the one parameter family defined by $xy + tz^2$, for $t \in \mathbb{C}$, the conic C_t is smooth for $t \neq 0$ and a pair of two different lines for $t = 0$. The corresponding map from $\mathbb{C}$ to $\{N, L, D\}$ sends $\mathbb{C} \smallsetminus 0$ to N and 0 to L; this shows that L must be in the closure of N. Similarly the family $x^2 + ty^2$, $t \in \mathbb{C}$, shows that D is in the closure of L. So we'd want the closed subsets of $\{N, L, D\}$ to be $\emptyset$, $\{D\}$, $\{D, L\}$, and $\{D, L, N\}$. This cannot be a fine moduli space, (nontrivial automorphisms of the conics prevent this), nor does it provide a course solution in algebraic geometry to the moduli problem. This illustrates the principle that the geometric points of a stack may tell us very little about it.

There a concrete description, familiar in algebraic geometry. One identifies the space of plane conics with the projective space $\mathbb{P}^5$ of homogeneous polynomials $ax^2 + bxy + cy^2 + dxz + eyz + fz^2$ of degree two in x, y, z modulo multiplication by nonzero scalars; so a plane conic defined by this polynomial is identified with the point $[a : b : c : d : e : f]$ in $\mathbb{P}^5$. The equation $ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$ defines a universal family $Y \subset \mathbb{P}^2 \times \mathbb{P}^5$ over $\mathbb{P}^5$. The group $G = PGL_3$ of projective linear transformations of $\mathbb{P}^2$ acts on the space of conics: an element of G defines an isomorphism $g \colon \mathbb{P}^2 \to \mathbb{P}^2$. For a conic Z in $\mathbb{P}^2$ the image $g(Z)$ is another conic, and the restriction $g|_Z \colon Z \to g(Z)$ is an isomorphism. Furthermore it is easy to see that two plane conics Z and W are isomorphic if and only if $W = g(Z)$ for a suitable $g \in G$, and that in this case the isomorphisms from Z to W are precisely the restrictions $h|_Z$ of those $h \in G$ such that $h(Z) = W$.[5] From this point of view, the moduli space of conics should be a quotient of $\mathbb{P}^5$ by the group G, and we may expect the moduli stack to be the quotient stack $[\mathbb{P}^5/G]$.

A categorical description of the stack of planar conics is a bit more complicated. A family of conics is a projective morphism $\pi \colon C \to S$, flat and of finite presentation, such that each geometric fiber is isomorphic to one of the three types of plane conics. Such a family comes with a $\mathbb{P}^2$-bundle $P \to S$, with C embedded into P as a closed subscheme[6]; locally, over an affine covering $\{U_\alpha\}$ of S, there are isomorphisms $P|_{U_\alpha} \cong \mathbb{P}^2 \times U_\alpha$ of $\mathbb{P}^2$-bundles, taking $C|_{U_\alpha}$ to the zeros of a degree 2 homogeneous polynomial which does not vanish identically at any point of U_α. If $E \to S$ is the bundle of local isomorphisms of P with $\mathbb{P}^2$, then E is a principal G-bundle over S, and we have a G-equivariant

[5] The group $G = PGL_{n+1}$ acts on the left on $\mathbb{P}^n$, so it acts on the right on the polynomials $\Gamma(\mathbb{P}^n, \mathcal{O}(m))$ of degree m by the formula $(F \cdot g)(x) = F(g \cdot x)$.

[6] In fact, P may be taken to be the projective bundle $\mathbb{P}(\mathcal{E})$ of lines in the rank 3 vector bundle $\mathcal{E} := \pi_*(\omega_{C/S}^\vee)$, where $\omega_{C/S}$ is the relative dualizing sheaf. A reader to whom this is unfamiliar can take this added structure of an embedding in a $\mathbb{P}^2$-bundle as part of the definition. Note that for a general base scheme S our notion of projective is that of [**EGA** II.5.5].

morphism from E to $\mathbb{P}^5$ that takes a point s to the image of $C_s \subset P_s$, which is a conic in $\mathbb{P}^2$, i.e., a point in $\mathbb{P}^5$. This pair $(E \to S, E \to \mathbb{P}^5)$ is an object of the category $[\mathbb{P}^5/G]$, and indicates why the stack of planar conics should be isomorphic to the quotient stack $[\mathbb{P}^5/G]$.

An important part of a moduli problem is to describe the automorphism groups of its objects. When the solution is a quotient by a group action, this is the same as describing the stabilizers of representative points. For conics, we have the three cases:

(1) N: $x^2+y^2+z^2$; the stabilizer consists of the complex orthogonal 3×3 matrices (i.e, those A such that ${}^tA \cdot A = I$, up to scalars . This group has dimension 3.

(2) L: $xy = 0$; the stabilizer consists of all invertible 3×3 matrices A modulo scalars, where A is of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \text{ or } \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ 0 & 0 & * \end{pmatrix}.$$

It has dimension 4.

(3) D: $x^2 = 0$; the stabilizer is the set of all matrices of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Its dimension is 6.

In general, if a smooth algebraic group G of dimension k acts freely on a smooth variety of dimension d, then the quotient (when it exists) will be smooth of dimension equal to $d-k$. This is not true in the naive world if the action is not free, but it will remain true in the stack world. A smooth orbit for the action will correspond to a "point" in the quotient and the dimension of this point will be equal to the dimension of the orbit minus k.

In the case of conics, $G = PGL_3$ has dimension 8, and the action of G on $\mathbb{P}^5$ has three orbits corresponding to the isomorphism types N, L, D of conics. The corresponding orbits are smooth of dimensions 5, 4 and 2 respectively. Therefore the quotient consists of three points: an open one, N, of dimension $5-8=-3$; in its closure another point L of dimension $4-8=-4$; and in its closure the point D of dimension $2-8=-6$. Note that these dimensions are precisely the negatives of the dimensions of the automorphism groups of conics in N, L and D respectively. (For an algebraic group H, the dimension of the stack $BH = [\bullet/H]$ will be $-\dim H$.)

As with triangles, the atlas we have given is only one of many. For example, one could take U to be the conics passing through the point $[0:0:1]$. This is a hyperplane in $\mathbb{P}^5$, defined by the vanishing of the coefficient of z^2. Take R to be the subset of $U\times G$ consisting of those pairs (u, g) such that $u \cdot g$ is also in U. There is a natural groupoid structure s, t, e, m, i on U and R so that the inclusion of U in $\mathbb{P}^5$ and the inclusion of R in $\mathbb{P}^5 \times G$ determines a morphism of groupoids from $R \rightrightarrows U$ to $\mathbb{P}^5 \times G \rightrightarrows \mathbb{P}^5$; this morphism satisfies Condition 1.3(i)–(ii). Note that this groupoid is not of the form $U \times H \rightrightarrows U$, for any action of a group H on U.

5. Elliptic curves

Elliptic curves have been a fruitful area for the development of moduli problems, as well as stacks (e.g., [**71**], [**21**]). We will devote Chapter 12 to elliptic curves, including the cases of arbitrary characteristic and over $\mathbb{Z}$. Here we sketch a few of the ideas, working in the category $\mathcal{S}$ of schemes over $\mathbb{C}$.

It is known classically that an elliptic curve E over $\mathbb{C}$ is classified up to isomorphism by a value $j \in \mathbb{C}$ known as the *j-invariant*, and all complex numbers occur. The (coarse) moduli space for isomorphism classes of elliptic curves should therefore be $\mathbb{C}$ (the complex plane, or, to an algebraic geometer, the affine line $\mathbb{A}^1$). However, the j-line is not a fine moduli space, as we will soon see; and, in fact, no fine moduli space exists.

A family of elliptic curves is a smooth and proper morphism $C \to S$, whose geometric fibers are connected curves of genus 1, together with a section $\sigma\colon S \to C$. We often abbreviate this data to $C \to S$, or sometimes even C. A morphism from $C' \to S'$ to $C \to S$ is pair of morphisms $C' \to C$ and $S' \to S$ such that the diagrams

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d] \\ C' \arrow[r] & C \end{tikzcd}$$

commute, and the first (and therefore the second) is cartesian. This determines the category $\mathcal{M}_{1,1}$, and the functor to $\mathcal{S}$ is the obvious one.

The section σ determines an origin in each fiber, which then gets the structure of an abelian variety; the section is called the zero section, or the identity section.[7] Note that every elliptic curve comes with an involution, written $p \mapsto -p$, that takes a point to its inverse with respect to this group structure. For instance, if $f(x)$ is a cubic polynomial over the complex numbers with 3 distinct roots, then $y^2 = f(x)$ is the equation of (an affine model of) an elliptic curve E. When $S = \operatorname{Spec}\mathbb{C}$ and C is the elliptic curve E, the identity section is the point at infinity, and the involution sends (x, y) to $(x, -y)$.

One reason that $\mathbb{A}^1$ is not a fine moduli space is that there are non-trivial families whose fibers (at closed points) are all isomorphic – so-called *isotrivial* families. The corresponding map from S to a moduli space would be constant, and, if the moduli space were fine, the family would have to be trivial.

Exercise 1.7. Fix a cubic polynomial $f(x)$ with 3 distinct roots, and let E be the elliptic curve defined by $y^2 = f(x)$. We take $S = \mathbb{A}^1 \smallsetminus \{0\}$, with coordinate t. Let $C \to S$ be the family of elliptic curves defined by the equation

$$ty^2 = f(x).$$

(1) Every fiber of this family is isomorphic to E.
(2) This family has only finitely many sections, hence is non-trivial.

Another reason that $\mathbb{A}^1$ cannot be a fine moduli space is that there are natural line bundles that one can obtain on a scheme S given any family of elliptic curves over

[7] In fact, the family gets the structure of a group scheme over S (see [**48**], §2).

S. One such line bundle assigns to a family $C \to S$ the normal bundle to the section $S \to C$. This assignment is natural in that sense that a morphism from $C' \to S'$ to $C \to S$ determines an isomorphism of the line bundle on S' with the pullback of the corresponding line bundle on S. (Such data, with an appropriate compatibility condition, is what is meant by a line bundle on the stack $\mathcal{M}_{1,1}$.) Such line bundles are not always trivial, as we will see. But there are no nontrivial (algebraic or analytic) line bundles on $\mathbb{A}^1$, so these line bundles cannot be pulled back via a morphism to $\mathbb{A}^1$.

One can study such line bundles without the formal language of stacks, and this is what Mumford did in [**71**]. He showed that there are exactly 12 such bundles (up to isomorphism), all tensor powers of the one just discussed. In stack language, this will say that $\mathrm{Pic}(\mathcal{M}_{1,1})$ is $\mathbb{Z}/12\mathbb{Z}$.

In this seminal paper, Mumford introduced the notion of a "modular family". This is a collection $\{\pi_\alpha\colon C_\alpha \to S_\alpha\}$ of families of elliptic curves, with the following property: each S_α must be a smooth curve, and, moreover, any first-order deformation of the fiber of π_α at $s \in S_\alpha$ must be captured by some tangent to S_α at s. The idea is that these $\{S_\alpha\}$ should be étale over the ideal moduli space (which cannot exist). This can be expressed by the assertion that, for any diagram

$$\begin{tikzcd} \mathrm{Spec}(\Gamma/I) \arrow[r] \arrow[d, hook] & S_\alpha \arrow[d] \\ \mathrm{Spec}(\Gamma) \arrow[r] \arrow[ur, dashed] & \mathcal{M}_{1,1} \end{tikzcd}$$

with Γ an Artin $\mathbb{C}$-algebra, I an ideal in Γ such that $I^2 = 0$, the dotted arrow can be filled in uniquely to make the diagram commute. Precisely, this says that for any family of elliptic curves $\pi\colon C \to \mathrm{Spec}(\Gamma)$, any morphism $\bar{f}\colon \mathrm{Spec}(\Gamma/I) \to S_\alpha$, and any isomorphism of families $\bar{\vartheta}\colon C \otimes_\Gamma \Gamma/I \to \mathrm{Spec}(\Gamma/I)\ {}_{\bar{f}}\!\times_{\pi_\alpha} C_\alpha$, there is a unique morphism $f\colon \mathrm{Spec}(\Gamma) \to S_\alpha$ lifting $\bar{f}$ and a unique isomorphism of families $\vartheta\colon C \to \mathrm{Spec}(\Gamma)\ {}_f\!\times_{\pi_\alpha} C_\alpha$ lifting $\bar{\vartheta}$.[8]

A modular family $\{\pi_\alpha\colon C_\alpha \to S_\alpha\}$ can be called a *covering* if every elliptic curve is isomorphic to some fiber of some π_α. (This makes $\{S_\alpha\}$ an étale covering of the nonexistent moduli space.)

We will see how a covering modular family determines an algebraic groupoid, which in fact is an atlas for the moduli stack $\mathcal{M}_{1,1}$.

To construct modular families, we need a few facts from the theory of elliptic curves, as found say in [**85**] when the base is a point, supplemented by [**19**] or [**48**] for families. Any elliptic curve can be embedded in the projective plane, with its chosen origin taken to the point $[0:1:0]$, and with an equation $y^2z = x^3 + Axz^2 + Bz^3$, with A and B complex numbers such that the form on the right vanishes at three distinct points in

[8]This condition makes S_α what is called a universal deformation space at each of its points. Note that the first-order deformations of an elliptic curve E are parametrized by $H^1(E, \mathcal{T}_E) = H^1(E, \mathcal{O}_E)$, which is 1-dimensional. That each S_α must be smooth and 1-dimensional therefore follows from the lifting property to be a modular family, as it identifies complete local rings on S_α at $\mathbb{C}$-points with universal deformation rings for the fibers of π_α.

the projective line. We write this in affine coordinates:

$$y^2 = x^3 + Ax + B, \qquad (4A^3 + 27B^2 \neq 0).$$

This is a family over $E \subset \mathbb{P}^2 \times W$ over $W = \{(A, B) \in \mathbb{C}^2 \mid 4A^3 + 27B^2 \neq 0\}$, called a *Weierstrass family*.[9] The j-invariant is given by[10]

$$j = 1728 \cdot \frac{4A^3}{4A^3 + 27B^2}.$$

All j-invariants (elliptic curves) occur in this family, but it is 2-dimensional, so it cannot be a modular family. We will look at some 1-dimensional families by restricting this to various lines, first a diagonal line, and then a horizontal and a vertical line.

The family

$$C_0 : \qquad y^2 = x^3 + \frac{27}{4} \cdot \frac{j}{1728 - j}(x + 1)$$

over $S_0 = \mathbb{A}^1 \smallsetminus \{0, 1728\}$ has the virtue that the j-invariant of the fiber over j is j. However, this family cannot be extended smoothly across the two deleted points. In fact, any modular family that contains a curve with j-invariant 0 or 1728 must have curves with nearby j-invariants appearing multiple times. This is a hint of the "stackiness" of this moduli problem: 0 and 1728 are precisely the j-invariants of the elliptic curves which possess additional automorphisms besides the identity and the involution $p \mapsto -p$.

Consider the family $C_1 \to S_1$ with

$$C_1 : \qquad y^2 = x^3 + Ax + 1$$

and $A \in S_1 = \{A \in \mathbb{A}^1 \mid 4A^3 + 27 \neq 0\}$. This attains every j-invariant except $j = 1728$. The family $C_2 \to S_2$ with

$$C_2 : \qquad y^2 = x^3 + x + B$$

with $B \in S_2 = \{B \in \mathbb{A}^1 \mid 4 + 27B^2 \neq 0\}$ attains every j-invariant but $j = 0$.

Exercise 1.8. These two families satisfy the conditions to be modular families.

Together, these two families form a covering modular family.

Exercise 1.9. Show that the morphism from $S_1 \amalg S_2$ to the affine line given by the j-invariant is unramified except over 0 and 1728, and show that the ramification index is 3 over $j = 0$ and 2 over $j = 1728$.

To make an atlas, we want to glue S_1 and S_2, and we must keep track of where an elliptic curve appears in both families. In the stack world, we don't just take an equivalence relation on $S_1 \amalg S_2$; rather, we keep track of automorphisms. That is, for α and β in $\{1, 2\}$, we consider the scheme $R_{\alpha,\beta}$ that parametrizes isomorphisms between C_α and C_β. Loosely speaking,

$$R_{\alpha,\beta} = \{(u, v, \phi) \mid u \in S_\alpha, v \in S_\beta, \text{ and } \phi \colon (C_\alpha)_u \xrightarrow{\simeq} (C_\beta)_v\}.$$

[9] In fact, any family $C \to S$ of elliptic curves is, locally in the Zariski topology, isomorphic to the pullback of $E \to W$ by a morphism from S to W.

[10] In [**71**] Mumford replaces j by $1728 - j$.

There are projections $s\colon R_{\alpha,\beta} \to S_\alpha$, taking (u, v, ϕ) to u, and $t\colon R_{\alpha,\beta} \to S_\alpha$, taking (u, v, ϕ) to v. Define

$$U = S_1 \amalg S_2$$

and take R to be the disjoint union of these four $R_{\alpha,\beta}$:

$$R = R_{1,1} \amalg R_{1,2} \amalg R_{2,1} \amalg R_{2,2}.$$

Then we have maps s and t from R to U. The multiplication m comes by composing the isomorphisms, taking $(u, v, \phi) \times (v, w, \psi)$ to $(u, w, \psi \circ \phi)$. The identity e takes $u \in S_\alpha$ to (u, u, id), where id is the identity map on $(C_\alpha)_u$; and the inverse i takes (u, v, ϕ) to (v, u, ϕ^{-1}). It is a straightforward exercise to verify that this forms a groupoid $R \rightrightarrows U$. This will be an atlas for the stack $\mathcal{M}_{1,1}$.

If two elliptic curves are given in Weierstrass form, $y^2 = x^3 + Ax + B$ and $y^2 = x^3 + A'x + B'$, it is a general fact that any isomorphism between them must be of the form $(x, y) \mapsto (\lambda x, \mu y)$ for some λ, $\mu \in \mathbb{C}^*$ (see [**85**], §III.3, and see [**19**], §1 for the version in families). So, for instance, we can express $R_{1,1}$ as the scheme consisting of all $\{(A, A', \lambda, \mu)\}$ such that

$$\mu^2x^3 + \mu^2Ax + \mu^2 = \lambda^3x^3 + \lambda A'x + 1.$$

In particular, $\mu^2 = 1$ and $\lambda^3 = 1$; setting $\gamma = \mu/\lambda$ we have $A' = \gamma^4A$ and γ can be any sixth root of unity. Let ϕ_γ denote the map $(x, y) \mapsto (\gamma^2x, \gamma^3y)$. Then

$$R_{1,1} \cong S_1 \times \mu_6$$

by associating $(A, \gamma^4A, \phi_\gamma)$ in $R_{1,1}$ to (A, γ) in $S_1 \times \mu_6$.

Exercise 1.10. Deduce, in a similar fashion, that $R_{2,2}$ is isomorphic to $S_2 \times \mu_4$ and that $R_{1,2}$ and $R_{2,1}$ can each be identified with the complement of 13 points in the affine line. (In fact the isomorphisms of curves can all be expressed conveniently in terms of the ϕ_γ.)

We want to see how this groupoid $R \rightrightarrows U$ can tell us about moduli of elliptic curves, i.e., about the category $\mathcal{M}_{1,1}$. We have a family $C \to U$, with $C = C_1 \amalg C_2$, and this contains every elliptic curve at least once. For any S and any map $\phi\colon S \to U$, we can pull back this family $C \to U$ to get a family on S, namely $C \times_U S \to S$. However, this fails two basic criteria to be a universal family:

(1) Two different maps $\phi_1\colon S \to U$ and $\phi_2\colon S \to U$ may determine isomorphic families on S.
(2) Some families over S may not be pullbacks from any morphisms from S to U.

As far as (1) is concerned, an isomorphism from the first pullback to the second determines (and is determined by) a morphism $\psi\colon S \to R$ that takes a point s to the given isomorphism from $C_{\phi_1(s)}$ to $C_{\phi_2(s)}$. In short, we have

$$\psi\colon S \to R \quad \text{with} \quad s \circ \psi = \phi_1 \quad \text{and} \quad t \circ \psi = \phi_2.$$

An extreme example of this occurs with $S = R$, $\phi_1 = s$, $\phi_2 = t$, in which case ψ is the identity.

Taking $S = S_0 = \mathbb{A}^1 \smallsetminus \{0, 1728\}$, the family $C_0 \to S$ is an example of the failure of (2): it is not the pullback from any map from S to U. However, it is *locally* a pullback: near any j in S, there is a disk Δ containing j and a morphism $\Delta \to U$ so that the restriction of the family to Δ is isomorphic to the pullback of the family $C \to U$. This works in the analytic category, but not in the algebraic category, if one uses the Zariski topology. Indeed, the only nonempty Zariski open sets in S are the complements of finite sets. But one can find a variety S', with a surjective morphism $\rho\colon S' \to S$, which is locally an analytic isomorphism — this makes it *étale* — together with a morphism $\phi\colon S' \to U$, with an isomorphism ϑ from the pullback of $C_0 \to S$ to S' via ρ with the pullback of $C \to U$ via ϕ.

Exercise 1.11. Show that $S' = \{\, a \in \mathbb{A}^1_{\mathbb{C}} \mid a \neq 0,\ 4a^6 + 27 \neq 0 \,\}$ is such a variety (with family $y^2 = x^3 + a^2x + 1$), and with the map $\rho\colon S' \to S$ given by $a \mapsto 1728 \cdot 4a^6/(4a^6 + 27)$, and $\phi\colon S' \to S_1 \subset U$ given by $a \mapsto A(a) = a^2$.

For such a "covering" $\rho\colon S' \to S$ (or S' a disjoint union of disks in the analytic case), we have a groupoid $S' \times_S S' \rightrightarrows S'$. For any point (s', s'') in $S' \times_S S'$, with s their common image in S, we have isomorphisms $C_{s'} \cong (C_0)_s \cong C_{s''}$.

Exercise 1.12. Show that these fiberwise isomorphisms are given by a (unique) global isomorphism of $(\phi \circ p_1)^*(C)$ with $(\phi \circ p_2)^*(C)$ on $S' \times_S S'$. This defines a morphism Φ from $S' \times_S S'$ to R with $s \circ \Phi = \phi \circ p_1$ and $t \circ \Phi = \phi \circ p_2$. Show that (ϕ, Φ) determines a morphism from the groupoid $S' \times_S S' \rightrightarrows S'$ to the groupoid $R \rightrightarrows U$.

Note that $S' \times_S S' \rightrightarrows S'$ is an atlas for S, and $R \rightrightarrows U$ is supposed to be an atlas for $\mathcal{M}_{1,1}$, so the groupoid morphism of the exercise can be regarded as a geometric realization of the morphism from (the stack corresponding to) S to the stack $\mathcal{M}_{1,1}$.

This picture can be reversed. Given an étale surjective map $\rho\colon S' \to S$, and a morphism (ϕ, Φ) from $S' \times_S S' \rightrightarrows S'$ to the groupoid $R \rightrightarrows U$, one gets a family $\phi^*(C)$ of elliptic curves on S' and an isomorphism $p_1^*(\phi^*C) \xrightarrow{\simeq} p_2^*(\phi^*C)$ on $S' \times_S S'$, satisfying a compatibility identity on $S' \times_S S' \times_S S'$. It is the theory of *descent* that implies that such a family is the pullback of a family on S (that is moreover unique up to unique isomorphism).

This example contains another fundamental insight of Grothendieck: Zariski open coverings $\{V_\alpha\}$ of a variety or scheme S should be replaced not just by $\coprod V_\alpha \to S$, but by arbitrary collections of étale morphisms $V_\alpha \to S$ whose (Zariski open) images cover S. When the base category is a category of schemes, the topology we will usually use — the *étale topology* — has these étale maps as its basic open sets.

Also by the theory of descent, the determination of $\mathrm{Pic}(\mathcal{M}_{1,1})$ can be reduced to the concrete computation of the the group of line bundles L on U equipped with isomorphisms $\varphi\colon s^*L \xrightarrow{\simeq} t^*L$ on R such that φ satisfies a natural compatibility condition on $R\ {}_t\!\times_s R$, up to isomorphism of such pairs (L, φ). The latter group is described by a finite amount of data: U has only trivial line bundles (since its components are Zariski open subsets of the affine line), and an isomorphism $s^*L \xrightarrow{\simeq} t^*L$ then given by an invertible function on R. So, $\mathrm{Pic}(\mathcal{M}_{1,1})$ is the quotient of the group of elements of $\mathcal{O}^*(R)$ satisfying the compatibility condition on $R\ {}_t\!\times_s R$ by the subgroup of elements of the form $t^*\chi/s^*\chi$, with $\chi \in \mathcal{O}^*(U)$. A tedious calculation yields an isomorphism $\operatorname{Pic}\mathcal{M}_{1,1} \cong \mathbb{Z}/12\mathbb{Z}$. This calculation will be carried out (using slightly different atlases) in Chapter 12.

In the analytic category, one has a modular family $E \to \mathbb{H}$ of elliptic curves over the upper half plane $\mathbb{H}$ whose fiber over τ in $\mathbb{H}$ is the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$, with Λ_τ the lattice $\mathbb{Z} + \mathbb{Z}\cdot\tau$. An isomorphism from E_τ to $E_{\tau'}$ is given by multiplication by a unique complex number ϑ such that $\vartheta \cdot \Lambda_\tau = \Lambda_{\tau'}$. A corresponding atlas is the groupoid $R \rightrightarrows \mathbb{H}$, where $R = \{(\tau, \tau', \vartheta) \in \mathbb{H} \times \mathbb{H} \times \mathbb{C} \mid \vartheta \cdot \Lambda_\tau = \Lambda_{\tau'}\}$. In fact, Mumford uses this analytic modular family in [**71**] to give a calculation of $\operatorname{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}$ in the analytic category.

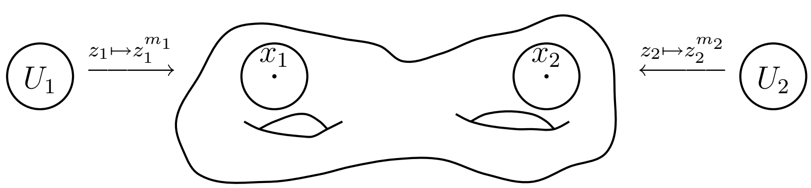
Exercise 1.13. Show that this analytic groupoid $R \rightrightarrows \mathbb{H}$ is isomorphic to the transformation groupoid $SL_2(\mathbb{Z}) \ltimes \mathbb{H}$ coming from the standard action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

6. Orbifolds

Orbifolds, sometimes called V-manifolds, provide another good introduction to some of the notions involved with stacks. In fact, the moduli stack of triangles, or any situation where a finite group acts on a manifold, gives rise to an orbifold. An orbifold is often described as a space that is locally a quotient of a manifold by a finite group, but this description is too crude: to give an orbifold, one must describe these local group actions, at least up to some equivalence. We will see that this extra data amounts to the difference between an ordinary space and a stack. (In fact, the underlying space corresponds to the coarse moduli space of the stack.)

As a simple example, let X be a Riemann surface, and let $x_1, \ldots, x_n$ be a finite set of points of X, and let $m_1, \ldots, m_n$ be positive integers, each greater than or equal to 2. Take a neighborhood V_i of x_i biholomorphic to a disk, and choose an isomorphism $V_i \cong U_i/G_i$, where U_i is a disk, and G_i is the cyclic group of m_i^{th} roots of unity, acting by rotation; take all the neighborhoods V_i to be disjoint.



At any other point x of X, choose any neighborhood of x biholomorphic to a disk and not containing any of the points x_i. These data determine an orbifold structure on the Riemann surface. Although the underlying (coarse) space is the original surface X, the orbifold structure is different, at any point x_i with $m_i > 1$. (See [**69**] for more on these Riemann surface orbifolds.)

For an explicit example, let $X = S^2 = \mathbb{C} \cup \{\infty\}$, with one point $p_1 = \infty$, and with $m_1 = m$. This is sometimes called the *m-teardrop.*

We turn next to a precise definition of an orbifold, following Haefliger [**43**], §4. (Compare Kawasaki's variation [**49**] of Satake's original [**80**].) We will define a complex analytic orbifold, although similar constructions work in other categories, cf. [**70**].

One starts with a topological space X. The data to give an orbifold structure to X consists of an open covering $\{V_\alpha\}$ of X, together with homeomorphisms $V_\alpha \cong G_\alpha \backslash U_\alpha$, where U_α is a connected complex manifold (usually taken to be an open set in $\mathbb{C}^n$), G_α is a finite group of analytic automorphisms of U_α, and $G_\alpha \backslash U_\alpha$ denotes the set of orbits, with the quotient topology inherited from U_α. (The action of G_α on U_α is assumed to be effective, i.e., $G_\alpha \subset \mathrm{Aut}(U_\alpha)$.) This data must satisfy the following compatibility condition: if $u \in U_\alpha$, and $u' \in U_\beta$ map to the same point in X, there must be neighborhoods W of u in U_α, and W' of u' in U_β, and a complex analytic isomorphism $\varphi\colon W \to W'$ taking u to u' and commuting with the projections to X:

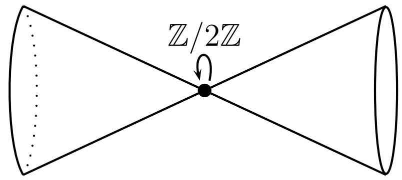
$$\begin{array}{ccccc} U_\alpha \supset W & \xrightarrow{\varphi} & W' \subset U_\beta \\ \searrow & & \swarrow \\ & V_\alpha \subset X \supset V_\beta & \end{array}$$

Note that these germs are not part of the data for the orbifold, and they need not be unique; rather, their existence is a condition on the data. The same idea defines when two such data are compatible: the orbifold structure defined by the open covering $\{V'_\alpha\}$ with $V'_\alpha \cong G'_\alpha \backslash U'_\alpha$ is compatible if, whenever $u \in U_\alpha$ and $u' \in U'_{\alpha'}$ map to the same point in X, there exist neighborhoods of each point and a complex analytic isomorphism commuting with the projections to X. An **orbifold structure** on X is an equivalence class of orbifold data, where compatible data are called equivalent.

Each of the quotients $G_\alpha \backslash U_\alpha$ has the structure of complex analytic space in which the analytic functions on $V \subset V_\alpha$ are precisely the G_α-invariant analytic functions on the pre-image of V in U_α (cf. [**17**], §4). The structure sheaves of the V_α can be patched canonically so that X inherits a complex analytic structure; it will be the "coarse" space for the corresponding stack. If X is connected, the manifolds U_α all have the same dimension, called the *dimension* of the orbifold.

Given orbifold data on X, one can construct an analytic groupoid, as follows. Set $U = \coprod U_\alpha$, and set R to be the set of triples (u, u', φ), where u and u' are points in U with the same image in X, and φ is a germ of an isomorphism from a neighborhood of u to a neighborhood of u' over X. This R has a unique topology so that the two projections s and t from R to U (taking (u, u', φ) to u and u' respectively) are local homeomorphisms; this gives R the structure of a complex manifold. The other maps are easily defined: $e\colon U \to R$ takes u to (u, u, id), $i\colon R \to R$ takes (u, u', φ) to (u', u, φ^{-1}), and $m\colon R\; {}_t\!\times_s R \to R$ takes $(u, u', \varphi) \times (u', u'', \psi)$ to $(u, u'', \psi \circ \varphi)$. We will be able to regard an orbifold as a stack by means of this atlas.

The simplest example of orbifold is a finite group quotient. Here U is a manifold, G is a finite group with an effective action on U, and V is the quotient space $G \backslash U$. In this case R can be identified with $G \times U$ and we recover the transformation groupoid $G \ltimes U$. For instance, if $U = \mathbb{C}^2$ and $G = \mathbb{Z}/2\mathbb{Z}$, with the action of its generator given by $(x, y) \mapsto (-x, -y)$, then the quotient (analytic) space V is a quadric cone, isomorphic to the locus in $\mathbb{C}^3$ defined by the equation $uv = w^2$. In this case the orbifold quotient can be pictured as follows.



At the vertex of the cone there is a nontrivial orbifold structure (indicated by the arrow); the complement is a manifold.

EXERCISE 1.14. Construct a groupoid for the m-teardrop. Take $U = U_1 \coprod U_2$, with $U_1 = \mathbb{C}$ and $U_2 = D$ an open disk mapping to a neighborhood of ∞ by $z \mapsto 1/z^m$. Compute R, and s, t, m, e, and i.

Note that the canonical map from R to $U \times U$ is never injective, unless all the maps $U_\alpha \to X$ are local homeomorphisms, in which case X is a manifold with its trivial orbifold structure.

EXERCISE 1.15. For any u in U, the automorphism group $\mathrm{Aut}(u) = s^{-1}(u) \cap t^{-1}(u)$ is canonically isomorphic to the isotropy group $(G_\alpha)_u = \{g \in G_\alpha \mid g \cdot u = u\}$, if u is in U_α. The canonical morphism $R \to U \times U$ is injective if and only if all the isotropy groups are trivial.

For a point u in U_α, write G_u for the isotropy group $(G_\alpha)_u = \{g \in G_\alpha \mid g \cdot u = u\}$. Given one germ φ from u to u', over a point x in X, with $u \in U_\alpha$ and $u' \in U_\beta$, the other possible germs have the form $\varphi \circ g$, where g is in the isotropy group G_u; they also have the form $g' \circ \varphi$ for g' in $G_{u'}$. Fixing one such φ determines an isomorphism from G_u to $G_{u'}$, sending g to g' when $\varphi \circ g = g' \circ \varphi$. This means that one can assign an *isotropy group* G_x for each point x in X, defined to be G_u for any point u that maps to x. This group is determined only up to (inner) isomorphism, since changing φ gives another isomorphism of G_u with $G_{u'}$ differing by an inner automorphism. In fact, the map $g \mapsto g_*$, where g_* is the induced endomorphism of the tangent space $T_u U_\alpha \cong \mathbb{C}^n$, gives an embedding $G_u \hookrightarrow GL_n(\mathbb{C})$ (see [**17**], §4), so we have an embedding of G_x in $GL_n(\mathbb{C})$, unique up to conjugacy.

It is a general fact (cf. [**80**], p. 475), that any connected orbifold can be written globally as a quotient of a manifold M by a Lie group G, in fact, with $G = GL_n(\mathbb{C})$ in this complex case, with n the dimension of the orbifold. Let us work this out in the language of groupoids. Let $P_\alpha \to U_\alpha$ be the bundle of frames, with fiber over $u \in U_\alpha$ being the set of bases of the tangent space $T_u U_\alpha$. This is a principal right G-bundle, with action of $g = (g_{ij})$ on a frame $v = (v_1, \ldots, v_n)$ by $(v \cdot g)_i = \sum_j v_j g_{ji}$. The group G_α acts on the left on P_α, by $(\tau \cdot v)_i = \tau_*(v_i)$. This action is free, and commutes with the action of G. Therefore the quotient $M_\alpha = G_\alpha \backslash P_\alpha$ is a manifold, and G acts in the right on M_α. Let $\rho_\alpha \colon M_\alpha \to V_\alpha$ be the canonical projections. The orbifold data determine gluing maps from $\rho_\alpha^{-1}(V_\alpha \cap V_\beta)$ to $\rho_\beta^{-1}(V_\alpha \cap V_\beta)$, taking the class of a frame v to the class of the frame $\varphi_*(v)$, for any choice of local germ φ. These gluing data commute with the action of G, so we obtain a manifold M with a right action of $G = GL_n(\mathbb{C})$, and a projection from M to X that is constant on orbits.

To say that the orbifold is the same as the quotient $[M/G]$, we should compare the groupoid $M \times G \rightrightarrows M$ with the groupoid $R \rightrightarrows U$ defining the orbifold structure.

Exercise 1.16. Let $P = \coprod_\alpha P_\alpha$, with canonical projection $\pi \colon P \to U$. Let $Q = \{(v, v', \varphi) \mid v, v' \in P, \varphi \text{ a germ from } \pi(v) \text{ to } \pi(v')\}$. (a) Construct a groupoid $Q \rightrightarrows P$, with s and t taking (v, v', φ) to v and v' respectively, and $m((v, v', \varphi), (v', v'', \psi)) = (v, v'', \psi \circ \varphi)$. (b) Construct a morphism from $Q \rightrightarrows P$ to $R \rightrightarrows U$, taking (v, v', φ) to $(\pi(v), \pi(v'), \varphi)$, and verify that it satisfies Conditions 1.3(i)–(ii). (c) Construct a morphism from $Q \rightrightarrows P$ to $M \times G \rightrightarrows M$, taking (v, v', φ) to (v, g), where g is determined by the equation $v'_i = \sum_j \varphi_*(v_j) g_{ji}$, and show that this morphism satisfies the same two conditions.

The local charts on an orbifold are used to do analysis (see [**7**] and [**34**]). For example, a differential form is given by a compatible collection of G_α-invariant differential forms ω_α on U_α. In terms of the groupoid, this is a differential form ω on U such that $s^*(\omega) = t^*(\omega)$ on R. In fact, groupoids provide a useful setting for much of the study of orbifolds (see [**33**]).

It should perhaps be pointed out that some authors also use a more restricted notion of orbifold, where the groups G_α are not allowed to include any complex reflections (i.e. isomorphisms conjugate to those of the form $(z_1, \ldots, z_n) \mapsto (\zeta z_1, \ldots, z_n)$, where ζ is a root of unity, cf. [**78**]); in this case the coarse space X actually determines the orbifold. This rules out orbifold structures like the one we gave on a Riemann surface at the beginning of this section, however. We have seen a similar phenomenon for elliptic curves, where the j-line is a coarse moduli space, but the stack "remembers" the automorphisms of the elliptic curves.

The definition we have given here works also for differentiable or topological orbifolds, by replacing the word "complex analytic" by "differentiable" or "continuous", cf. [**70**]. One can give a corresponding definition in algebraic geometry, although here one must use étale neighborhoods to describe a notion of germ of an isomorphism. There are more general notions of orbifolds, cf. [**84**], where it is not required that the action of each G_α on U_α be effective. Both of these notions can be described more easily in the language of stacks.

7. Schemes, Functors, and Stacks

Before stacks, an approach to the study of families of algebraic objects was to consider a contravariant functor h from the base category of schemes $\mathcal{S}$ to the category of sets, with $h(S)$ being the set of isomorphism classes of families over S. For example, for $\mathcal{M}_g$, $h(S)$ was the set of families of curves $C \to S$, modulo isomorphism. The functor h is *representable* if there is a scheme X such that the functor h is naturally isomorphic to the functor of points (see Example 1.1A) h_X. One of the best known and most important examples of this is the functor that assigns to a scheme S the set of closed subschemes of $\mathbb{P}^n \times S$, flat over S; this is represented by a Hilbert scheme [**37**]. Most such functors, such as the one for moduli of curves, are not representable.

This approach is consistent with Grothendieck's idea of identifying a scheme with its functor of points (see Example 1.1A). Schemes, which generalize algebraic varieties, sit inside a larger category of (certain) functors, the top line in the following diagram of algebraic objects:

$$\begin{aligned}\text{Algebraic varieties} \subset \text{Schemes} \subset \text{Algebraic spaces}& \\ \subset \text{Deligne–Mumford stacks} \subset \text{Artin stacks}&\end{aligned}$$

Though they won't play a such a major role in this book, *algebraic spaces* are these functors [**4**] [**56**]; they form a class of algebraic objects which generalize schemes.[11]

The examples in this chapter have emphasized the point that geometric problems can lead to categories. These make up the bottom line of the diagram, the *algebraic stacks*. There are two different sets of axioms which make categories suitable for doing geometry. The focus of Part I of this book will be on *Deligne–Mumford stacks*, introduced by Deligne and Mumford in their stack-based proof of the irreducibility of moduli spaces of curves of genus g over arbitrary base fields [**20**]. In Part II of this book we will meet the more general *Artin stacks* [**5**].

Stabilizer groups were an important feature in the discussion surrounding the examples. In moduli problems, the stabilizer groups are the automorphism groups of the objects being parametrized. The main novelty of stacks, as opposed to varieties, schemes, or spaces, is the presence of nontrivial stabilizer groups (in fact, it will be shown that an algebraic stacks with no nontrivial stabilizers must be isomorphic to a scheme or an algebraic space). The distinction between Deligne–Mumford stacks and Artin stacks lies in the kind of stabilizer groups that are permitted. The stabilizer group at a geometric point of a Deligne–Mumford stack is always a finite group, whereas an Artin stack may have an arbitrary algebraic group (finite-type group scheme) as a geometric stabilizer. Stabilizer groups thus provide the answer to the question, "What makes something a stack and not a scheme or an algebraic space?"

Many of the examples presented in this chapter will end up being algebraic stacks. The stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}_g}$ are famous examples of Deligne–Mumford stacks. Stacks $[X/G]$, described in Example 1.1B in a topological setting, will be algebraic stacks when X is a scheme and G is an algebraic group; they are Artin stacks in general, and whether they are Deligne–Mumford stacks depends on what sorts of stabilizer groups they have. Conics form an Artin stack (positive-dimensional stabilizer groups), while $\mathcal{M}_{1,1}$ is an important example of a Deligne–Mumford stack (finite stabilizer groups). If a nonsingular complex variety is endowed with an orbifold structure, then it gives rise to a Deligne–Mumford stack. We will come back to these examples repeatedly throughout this book. Especially in the early chapters, this core collection of examples will serve as a counterbalance to the abstractions required in order to reach an answer to the question, "What is an algebraic stack?"

[11] There is actually a condition for a scheme to be an algebraic space: it must be *quasi-separated*, i.e., have quasi-compact diagonal (see the Glossary). This is really only a technical condition, since the schemes that one meets in practice are always quasi-separated.

Answers to Exercises

1.1. One axiom states that the composition of two arrows has the source of the first arrow as source and the target of the second arrow as target: $s \circ m = s \circ \mathrm{pr}_1$ and $t \circ m = t \circ \mathrm{pr}_2$ as maps $R\ {}_t\!\times_s R \to U$.

1.2. (a) $D \times S^1 \to X$, $(z, e^{i\vartheta}) \mapsto (z, \vartheta)$ is a 2-sheeted covering map, so satisfies the surjectivity requirement. (b) On X the group $\mathrm{Aut}(x)$ is trivial for x not on the central line, and it is $\{\pm 1\}$ for $x = (0, \vartheta)$.

1.3. The key fact is that $\widetilde{Y} \to Y$ and $\widetilde{T} \to T$, restricted to the pre-images of T°, are local homeomorphisms. With this, one sees that $Y^\circ \to T^\circ$ is a family of triangles. Any family of triangles with sides of distinct lengths is fiberwise uniquely identified with the pullback of $Y^\circ \to T^\circ$. That this gives a homeomorphism of families can be checked locally; locally we can make a choice of ordering of the vertices and argue as in the case of the universality property for $\widetilde{Y} \to \widetilde{T}$.

1.4. The commutative diagram

$$\begin{tikzcd}
 & (U_\alpha \cap U_\beta \cap U_\gamma)\ {}_{\phi_\alpha}\!\times_V Z \arrow[dd, "{(p,\Phi_{\beta\alpha}(p)\cdot z)}"] \arrow[dr, "{(p,\Phi_{\gamma\alpha}(p)\cdot z)}"] & \\
 & & (U_\alpha \cap U_\beta \cap U_\gamma)\ {}_{\phi_\gamma}\!\times_V Z \\
 & (U_\alpha \cap U_\beta \cap U_\gamma)\ {}_{\phi_\beta}\!\times_V Z \arrow[ur, "{(p,\Phi_{\gamma\beta}(p)\cdot z)}"] & \\
X|_{U_\alpha \cap U_\beta \cap U_\gamma} \arrow[uuur, "\sim"] \arrow[ur, "\sim"] \arrow[uurr, bend right, "\sim"'] & &
\end{tikzcd}$$

yields the crucial identity $\Phi_{\gamma\alpha}(p) = \Phi_{\gamma\beta}(p) \cdot \Phi_{\beta\alpha}(p)$.

1.5. The map of groupoids from $G \times \widetilde{V} \times \mathfrak{S}_3 \rightrightarrows \widetilde{V}$ to $G \times V \rightrightarrows V$ is given by $\phi(\tilde{v}) = v$ and $\Phi(g, \tilde{v}, \pi) = (g, v)$, where $g \in G$ and $\tilde{v} \in \widetilde{V}$, with v the triangle having vertices $\tilde{v}$. Given $(g, v) \in G \times V$, a point in $\widetilde{V} \times \widetilde{V}$ lying over $(v, g \cdot v) \in V \times V$ is determined by choosing an ordering $\tilde{v}$ of the vertices of v, and a re-ordering $\pi \in \mathfrak{S}_3$ of $g \cdot \tilde{v}$, hence Condition 1.3(i) is fulfilled. The map $\widetilde{V} \to V$ is a topological covering map; hence Condition 1.3(ii) is satisfied by choosing the identity element (e, v) of $G \times V$.

1.6. For the category $\mathfrak{T}$, take the same objects, but for morphisms allow the induced maps on fibers to be isometries followed by homotheties (multiplications by a positive scalar). Replace $\widetilde{T}$ by its intersection with the plane $a + b + c = 1$, and enlarge G by allowing homotheties. The resulting stack has dimension 2.

1.7. Under the substitution $s^2 = t$, sections correspond to maps $g\colon \mathbb{A}^1 \smallsetminus \{0\} \to E$ satisfying $g(-s) = -g(s)$, but g must be a constant map, so sections are in bijective correspondence with 2-torsion points of E.

1.8. That $C_1 \to S_1$ is modular can be shown using the criterion on p. 17: for any c in Γ, denote by $\bar{c}$ the image in Γ/I. Any elliptic curve over Γ has the form $y^2 = x^3 + ax + b$ for some a and b in Γ (see [**19**], §2). An isomorphism of $y^2 = x^3 + \overline{A}x + 1$ with $y^2 = x^3 + \bar{a}x + \bar{b}$ over Γ/I has the form $(x, y) \mapsto (\bar{\lambda}x, \bar{\mu}y)$ for units $\bar{\lambda}$ and $\bar{\mu}$ in Γ/I; these satisfy the equations $\bar{\lambda}^3 = \bar{\mu}^2 = \bar{b}$ and $\bar{a}\bar{\lambda} = \bar{\mu}^2\overline{A}$. Using the fact that 2 and 3 are invertible in Γ, one verifies that there are unique liftings λ, μ and A of $\bar{\lambda}$, $\bar{\mu}$ and $\overline{A}$ with $\lambda^3 = \mu^2 = b$ and $a\lambda = \mu^2 A$. Then (λ, μ) determine the required isomorphism of $y^2 = x^3 + Ax + 1$ with $y^2 = x^3 + ax + b$ over Γ, as required. A similar argument applies to the second family, solving equations $\lambda^3 = \mu^2 = a\lambda$ and $b = \mu^2 B$ for λ, μ and B.

1.9. For S_1, $j = 1728 \cdot 4A^3/(4A^3 + 27)$, and this has ramification index 3 over $j = 0$.

1.10.

$$
\begin{aligned}
R_{1,1} &= \{(A, \lambda^2 A, \lambda, \mu) \mid A \in S_1, \lambda^3 = 1, \mu^2 = 1\};\\
R_{2,2} &= \{(B, \mu^2 B, \mu^2, \mu) \mid B \in S_2, \mu^4 = 1\};\\
R_{1,2} &= \{\rho^{-4}, \rho^6, \rho^2, \rho^3) \mid \rho \neq 0, \rho^{12} \neq -27/4\}.
\end{aligned}
$$

1.11. Note that $\rho^*(C_0)$ is the family $y^2 = x^3 + a^6 x + a^6$, and an isomorphism ϑ from $\rho^*(C_0)$ to $\phi^*(C)$ is given by $(\lambda, \mu) = (a^{-2}, a^{-3})$.

1.12. The isomorphism on $S' \times_S S'$ can be constructed as the composite

$$
\begin{tikzcd}
(\phi \circ p_1)^* C \arrow[d, equal] & (\rho \circ p_1)^* C_0 \arrow[r, equal] \arrow[d, equal] & (\rho \circ p_2)^* C_0 \arrow[d, equal] & (\phi \circ p_2)^* C \arrow[d, equal] \\
p_1^* \phi^* C & p_1^* \rho^* C_0 \arrow[l, "\sim"'] & p_2^* \rho^* C_0 \arrow[r, "\sim"] & p_2^* \phi^* C
\end{tikzcd}
$$

1.13. An isomorphism is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \tau \mapsto (\tau, \tau', \vartheta)$, where $\tau' = \frac{a\tau + b}{c\tau + d}$ and $\vartheta = \frac{1}{c\tau + d}$. Note that although $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the same action on $\mathbb{H}$, the sign is determined by ϑ.

1.14. $R_{1,1} \cong U$, $R_{1,2} \cong R_{2,1} \cong D \setminus \{0\}$, $R_{2,2} \cong \mathbb{Z}/m\mathbb{Z} \times D$, with (k, z) corresponding to $(z, e^{2\pi i k/m} z, e^{2\pi i k/m})$ in $R_{2,2}$. The product on $R_{2,2}$ takes $(k, z) \times (l, e^{2\pi i k/m} z)$ to $(k + l, z)$.

1.15. For any $(u, u, \varphi) \in \mathrm{Aut}(u)$, with $u \in U_\alpha$, the given φ extends to an automorphism of U_α given by the action of unique $g \in G_\alpha$, such that $g(u) = u$.

1.16. The morphism of groupoids from $Q \rightrightarrows P$ to $R \rightrightarrows U$ is given by $\pi \colon P \to U$ and $(v, v', \varphi) \mapsto (\pi(v), \pi(v'), \varphi)$, and Condition 1.3(i) is immediate. The morphism of groupoids from $Q \rightrightarrows P$ to $M \times G \rightrightarrows M$ is given by $v \mapsto [v]$ and $(v, v', \varphi) \mapsto ([v], g)$ with g such that $\varphi_*(v) \cdot g = v'$. To verify Condition 1.3(i): say $v \in P_\alpha$ and $v' \in P_\beta$ are frames over points with the same image $x \in X$. Fix a germ φ from $u := \pi(v)$ to $\pi(v')$; for any other germ ψ the germ $\varphi^{-1} \circ \psi$ extends uniquely to the action of some $h \in (G_\alpha)_u$, and the condition now follows from the fact that $(G_\alpha)_u$ acts freely on $\pi^{-1}(u)$ with quotient $\rho_\alpha^{-1}(x)$. Since $P \to U$ and $P \to M$ have local sections (one is a bundle projection, the other is a surjective local homeomorphism), Condition 1.3(ii) is satisfied in both cases.