Preface

The notion of stacks grew out of attempts to construct moduli spaces in contexts where ordinary varieties or schemes were deficient. They may be regarded as the next step in the development from affine and projective varieties to abstract varieties to schemes to algebraic spaces. However, even when moduli spaces have been constructed as projective varieties, whenever some objects being parametrized have nontrivial automorphisms the usual (coarse) moduli space is not a faithful reflection of the moduli problem; in such cases, the corresponding stack is more useful.

It is not easy to point to one moment when stacks appeared. Many of the ideas that led to stacks can be found in work of Grothendieck, especially [**38**]. These ideas were developed by Giraud [**31**], and in the work of Artin [**4**] and Knutson [**56**] on algebraic spaces, which were an earlier extension of the notion of schemes. More ideas toward stacks can be found in Mumford's article [**71**], which remains an excellent source of inspiration.

Deligne and Mumford defined a notion of algebraic stack in [**20**] — the concept now known as *Deligne–Mumford stack* — and used moduli stacks of curves to give an alternative and conceptually simpler proof of the main theorem of that paper: the moduli space of curves of genus g over any field is irreducible. Concise definitions were given in [**20**], but the basic theorems were only stated, with details promised for another publication that unfortunately has never appeared. Since then the language of stacks has been found to be useful in a variety of moduli and other problems. Except for a few fragments, such as the appendix of [**89**] and section I.4 of [**25**], there had been little foundational exposition of these ideas until the book [**61**] was published.

Artin [**5**] defined a more general notion of stack, which can be used for moduli problems of objects with infinite automorphism groups. These have become increasingly important in recent years, such as for moduli of vector bundles ([**61**] and Chapter 19) and the construction of intrinsic normal cones to singular varieties ([**10**] and Chapter 17). The book of Laumont and Moret-Bailly [**61**] provides a systematic treatment of stacks, with proofs of many of the main theorems in the subject. However, it presumes quite a sophisticated background, and proceeds at a high level of difficulty.

The aim of these notes is to give a leisurely and elementary introduction to stacks and some of their uses. Although the language is necessarily abstract, by proceeding from examples to theory, and keeping the examples as simple as possible, we hope to make the notes accessible to those uncomfortable with fancy terminology. One of our goals, in fact, is to prepare a reader for [**61**].

We do, however, assume the reader is familiar with basic notions about schemes, including definitions and basic properties of smooth, étale, and flat morphisms. Many of these notions are defined briefly in a glossary at the back of this book. Good references for most of this material are [**74**] and [**47**], perhaps beginning with [**24**] or [**64**] — with [**EGA**] as the definitive source.

In Part I we make several simplifying assumptions, which suffice to describe Deligne–Mumford stacks and algebraic spaces. Many interesting examples, such as moduli spaces of curves, can be worked out in this language. We give a more direct translation between the categorical and the atlas notions of stack than had been available in the literature before. We include descriptions of stacks of dimensions 0 and 1. We compute the Picard group of the stack of elliptic curves, over any field and over the integers (the latter a new result). We do not try to prove all general theorems about Deligne–Mumford stacks, but we do include enough so that we can carry out proofs of the assertions made in [**20**]. Algebraic spaces are defined and studied as special cases of Deligne–Mumford stacks; the reader need not know about them in advance.

In Part II we begin again, without the special assumptions, to define general algebraic (Artin) stacks. In particular, we collect here in one place the various properties that are scattered throughout Part I. In Part II we proceed in a more concise manner, hoping that the reader who has spent time in Part I will be prepared for this. Here we discuss some important examples of algebraic stacks, such as moduli stacks of vector bundles, and cone stacks. We conclude with chapters on sheaves and cohomology.

In the literature one finds many assertions that something is a stack. What is rare is to find justification for such an assertion, such as a verification that a proposed stack satisfies all or even some of the axioms to be a stack. (The most common reason given for why something is a stack is to point out why it is *not* a scheme!) We hope that giving a few such proofs, together with examples, will improve this situation.

At the foundations of the subject there are quite a few categorical constructions, and verifications involve checking that many diagrams commute. Although we have left many of these verifications as exercises, we do include a section of Answers at the end, where the reader can find many of these worked out.

We are all familiar with the idea that a geometric object can have an intrinsic definition as a ringed space, and that it can also be constructed from an atlas by gluing together simple local models. Stacks also have a dual nature. Their intrinsic nature, however, is not a space with some structure; rather, it is a *category*, together with a functor from this category to a base category (usually of schemes or other familiar geometric objects). Stacks can also be realized from a kind of atlas, called a *groupoid*, that consists of a pair of objects together with five morphisms in the base category. (In the familiar case of a manifold obtained by gluing open sets $\{U_\alpha\}$, the two objects are $\coprod U_\alpha$ and $\coprod U_\alpha \cap U_\beta$, and the morphisms express the usual compatibilities among the gluing data.) For stacks, however, the distance between their categorical nature and an atlas is considerably greater than that between a ringed space and an atlas in classical geometry — roughly speaking, the role of topology for ringed spaces has to be replaced by Grothendieck's descent theory — and it will take us a few chapters to work this out thoroughly. We begin in Chapter 1, however, with a brief description of these categorical and groupoid notions, and discuss a collection of examples, which we hope will prepare the reader for this journey.

One feature of this text, especially in Part I, is an emphasis on these atlases, or groupoids. We believe that working explicitly with many of them leads to a better feeling for stacks. Of course, just as with manifolds, describing a particular stack by an atlas is often not the most revealing way to study it. However, the existence of such an atlas is crucial, since it allows the extention of basic notions from algebraic geometry to stacks by putting appropriate geometric conditions on the schemes and the maps among them.

In addition to the introductory Chapter 1, we have included an Appendix C which studies groupoids of sets. These can be thought of as descrete groupoids (or stacks). They provide a model for the general theory, in which many of the constructions appear in a simple setting, without geometric complications. Beginners may want to spend some time with this appendix before reading the main text.

The text include many Examples and Exercises, which are used in similar ways; Exercises may require more work, and Examples are more likely to be referred to later. In both, we frequently omit phrases like "Show that", especially for routine verifications.

Note on terminology. The notion of groupoid and the word "groupoid" originated in the 1920's in algebra, and in the 1950's in geometry, mainly in the work of Ehresmann and Haefliger; they have continued to appear in many areas of mathematics, and today they play a role in noncommutative geometry, cf. [**18**]. A brief history can be found in [**16**]. When the spaces involved are discrete, this notion coincides with that of a (small) category in which all morphisms are isomorphisms. Category theorists have taken the word groupoid to *mean* a category with this property, and many others have followed this practice.[1] As groupoids in the original sense play a fundamental role here, we will use the word in its original meaning, so we will have algebraic groupoids in algebraic geometry, topological groupoids in topology, etc. If this is not confusing enough, however, categories in which all maps are isomorphisms also appear prominently in the development of stacks. We will say that a category *is* a groupoid if it has this property.

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Comments, corrections, suggestions, or contributions (say for other elementary topics that could be included) are eagerly solicited!

[1] The reader has probably heard the joke that a group is a category with one object in which all maps are isomorphisms. Few take this seriously — at least, no one mistakes an abelian group for some kind of abelian category. Unfortunately for the term groupoid, this joke has been taken seriously!

[Tentative] **CONTENTS**