Analysis Prelim January 2017

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Problem A

Let Γ be the curve

$$\{(x, y) \in \mathbb{R}^2 : y = f(x)\}$$

where f is a continuous function on the real line. Show that $m(\Gamma) = 0$, where m is two-dimensional Lebesgue-measure (area).
Hint: Cover Γ by rectangles and use uniform continuity.

Solution:
First, we will show that this is true when f is function on a closed interval $[a, b] \subset \mathbb{R}$.
Fix $\epsilon > 0$.
Since f is continuous on $\mathbb{R}$ and $[a, b] \subset \mathbb{R}$ is a compact subset, f is uniformly continuous on $[a, b]$. Thus, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
Without loss of generality, assume $\delta < |b - a|$: If it's not, then we can choose another suitable $0 < \delta_1 < |b - a|$ that will also work for this ϵ.
Let $n \in \mathbb{N}$ be the smallest natural number such that $n\delta > |b - a|$. Then there exists a partition of $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

such that $|x_{i+1} - x_i| < \delta$. So for any $z \in [x_i, x_{i+1}]$, $f(z) \in [f(x_i) - \epsilon, f(x_i) + \epsilon]$.
Let $\Gamma_{[a,b]}$ denote the graph of f over $[a, b]$. By this construction:

$$\Gamma_{[a,b]} \subset \bigcup_{i=1}^{n} ([x_{i-1}, x_i] \times [f(x_{i-1}) - \epsilon, f(x_{i-1}) + \epsilon])$$

By properties of Lebesgue measure:

$$\begin{aligned} m(\Gamma_{[a,b]}) &\leq m\left(\bigcup_{i=1}^{n} ([x_{i-1}, x_i] \times [f(x_{i-1}) - \epsilon, f(x_{i-1}) + \epsilon])\right) \\ &\leq \sum_{i=1}^{n} m([x_{i-1}, x_i]) \cdot m([f(x_{i-1}) - \epsilon, f(x_{i-1}) + \epsilon]) \\ &= \sum_{i=1}^{n} |x_i - x_{i-1}| \cdot |2\epsilon| \\ &< 2n\delta\epsilon \end{aligned} \tag{1}$$

Since n was chosen as the smallest natural number such that $n\delta > |b - a|$, we have $(n - 1)\delta \leq |b - a| \Rightarrow n\delta \leq b - a + \delta$. Plugging this information into (1):

$$\begin{aligned} m(\Gamma_{[a,b]}) &< 2n\delta\epsilon \\ &\leq 2\epsilon(b - a + \delta) \\ &\text{And since } \delta \leq b - a\text{:} \\ &\leq 4\epsilon(b - a) \end{aligned}$$

$b - a$ is fixed and ϵ can be made arbitrarily small, so this shows $m(\Gamma_{[a,b]}) = 0$.
This holds for any arbitrary closed interval $[a, b]$.

Let $\{x_n\}_{n=1}^{\infty}$ enumerate the integers. We can re-write Γ as a countable union of these sub-graphs:

$$\Gamma = \bigcup_{n=1} \Gamma_{[x_n, x_{n+1}]}$$

By the countable additivity of measure:

$$m(\Gamma) \leq \sum_{n=1} m(\Gamma_{[x_n, x_{n+1}]}) = \sum_{n=1}^{\infty} 0 = 0$$

$\square$

Problem B

Let E be a given set in $\mathbb{R}^2$ and let d denote the distance between a point and the set:

$$d(x, E) := \inf\{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, y \in E\}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.
Consider the open set

$$O_n := \{x : d(x, E) < 1/n\}$$

Show that

(1) If E is compact then $m(E) = \lim_{n\to\infty} m(O_n)$, where m is two-dimensional Lebesgue measure.

(2) If E is closed and unbounded or E is open and bounded, then the conclusion is false. **Hint:** In the second case, consider a dense countable set in the unit square.

Solution:

(1) If E is compact then it is closed and bounded. Since E is closed, we can show $E = \bigcap_{n=1}^{\infty} O_n$:

First, note that $\bigcap_{n=1}^{\infty} O_n = \{x : d(x, E) = 0\}$.
If $x \in \{x : d(x, E) = 0\}$, then for every $n \in \mathbb{N}$, there exists $y_n \in E$ such that $d(x, y_n) < 1/n$. $\{y_n\} \subset E$ converges to x, and since E is closed, $x \in E$.
This shows $E = \bigcap_{n=1}^{\infty} O_n$.
The O_n are nested sets: $O_1 \supseteq O_2 \supseteq \cdots$
By continuity of measure for intersections of nested sets:

$$m(E) = m\left(\bigcap_{n=1}^{\infty} O_n\right) = \lim_{n\to\infty} m(O_n)$$

(2) If E is closed and unbounded, this does not hold. Suppose $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$.
Then, $O_n = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1/n\}$. $m(O_n) = \infty$ for all n.
However, $m(E) = 0$. We will show this by covering E with rectangles whose total area is ϵ, for an arbitrary $\epsilon > 0$:
First, focus on the ray $[0, \infty) \times \{0\}$, first.
Cover $[0, 1] \times \{0\}$ with a rectangle width 1 and height $\epsilon \cdot 2^{-2}$. Cover $[1, 2] \times \{0\}$ with a rectangle width 1 and height $\epsilon \cdot 2^{-3}$...Cover $[n, n+1] \times \{0\}$ with a rectangle width 1 and height $\epsilon \cdot 2^{-n-2}$. The total area of these rectangles will be:

$$\epsilon \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\epsilon}{2}$$

Covering the ray $(-\infty, 0]$ in the same fashion, we get a covering of E by rectangles of total area ϵ. Since ϵ can be made arbitrarily small, $m(E) = 0$. Thus:

$$0 = m(E) \neq \lim_{n\to\infty} m(O_n) = \infty$$

If E is open and bounded, the equation still does not hold. Suppose E is the set of points (x, y) in the unit square $[0,1] \times [0,1]$ with $x, y \in \mathbb{Q}$.
Since the rationals are dense, this is a dense subset of $[0,1] \times [0,1]$.
Also $m(E) = 0$, because E is a countable set of points, and $m(x, y) = 0$.
However, any open set containing E necessarily contains $[0,1] \times [0,1]$, since E is dense in the unit square.
This means $[0,1] \times [0,1] \subseteq \bigcap_{n=1}^{\infty} O_n$.
$m([0,1] \times [0,1]) = 1$, by definition of Lebesgue measure. By countable additivity and the continuity of measure:

$$\begin{aligned} 1 &= m([0,1] \times [0,1]) \\ &\leq m\left(\bigcap_{n=1}^{\infty} O_n\right) \\ &= \lim_{n\to\infty} m(O_n) \end{aligned}$$

So $\lim_{n\to\infty} m(O_n) \geq 1$, and we do not have $m(E) = \lim_{n\to\infty} m(O_n)$.

Problem C

Prove that in a Hilbert space, if $|\langle f, g\rangle| = \|f\|\,\|g\|$ and $g \neq 0$, then $f = cg$ for some scalar c.
Hint: Consider the normalized vectors and use the Pythagoras theorem.

Solution:
The Cauchy-Schwarz Inequality immediately gives:

$$|\langle f, g\rangle| \leq \|f\|\,\|g\|$$

This holds for all f, g in the Hilbert space.
If $f = cg$:

$$\begin{aligned} |\langle f, g\rangle| &= |\langle cg, g\rangle| \\ &= |c||\langle g, g\rangle| \text{ (by linearity of inner product)} \\ &= |c|\,\|g\|^2 \\ &= \|cg\|\,\|g\| \\ &= \|f\|\,\|g\| \end{aligned}$$

Now, suppose $|\langle f, g\rangle| = \|f\|\,\|g\|$.
Set $x = \frac{|\langle f,g\rangle|}{\|g\|^2} g$.
This definition together with our hypothesis yields:

$$\|x\| = \|f\|$$

There exists a vector y such that $f = x + y$ and $x \perp y$.
By the Pythagorean Theorem:

$$\|f\|^2 = \|x\|^2 + \|y\|^2 \tag{2}$$

But we also have $\|x\| = \|f\|$, so this implies $\|y\| = 0$ and $f = x$.

□

Problem D

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous, and such that the Lebesgue integral

$$\int_{\mathbb{R}} f(x,s)dx$$

exists for all $s \in \mathbb{R}$. Assume that, for any $x \in \mathbb{R}$, $f(x,s)$ is differentiable everywhere with respect to s, and that, for some function $g \in L^1(\mathbb{R})$,

$$\left|\frac{\partial}{\partial s} f(x,s)\right| \leq g(x)$$

for all $(x,s) \in \mathbb{R} \times \mathbb{R}$. Show that you can differentiate under the integral sign: that is, show that

$$\frac{d}{ds}\int_{\mathbb{R}} f(x,s)dx = \int_{\mathbb{R}} \frac{\partial}{\partial s} f(x,s)dx$$

Do not use any theorems you may already know about differentiating under the integral sign. However, you may use results such as the Dominated Convergence Theorem or the Bounded Convergence Theorem. Also, the Mean Value Theorem may be of help.

Solution:
Fix $x \in \mathbb{R}$.
By the Mean Value Theorem, for any $a, b \in \mathbb{R}$, there exists $c \in (a,b)$ such that:

$$\frac{f(x,b) - f(x,a)}{b-a} = \frac{\partial f}{\partial s}(x,c)$$

Since

$$\left|\frac{\partial}{\partial s} f(x,s)\right| \leq g(x),$$

we can conclude that

$$\left|\frac{f(x,b) - f(x,a)}{b-a}\right| = \left|\frac{\partial f}{\partial s}(x,c)\right| \leq g(x)$$

Let $\{a_i\}$ be a sequence which converges to s, with $a_i \neq s$ for all i. Since $\mathbb{R}$ is complete, this exists for all $s \in \mathbb{R}$.
Then, look at the sequence:

$$\left\{\frac{f(x,s) - f(x,a_i)}{s - a_i}\right\}_{i=1}^{\infty}$$

This sequence is bounded above by $g(x)$ (using the work above), so by the Lebesgue Dominated Convergence Theorem, we can switch the limit and the integral:

$$\int_{\mathbb{R}} \frac{\partial}{\partial s} f(x,s)dx = \int_{\mathbb{R}} \lim_{i\to\infty} \frac{f(x,s) - f(x,a_i)}{s-a_i}dx = \lim_{i\to\infty}\int_{\mathbb{R}} \frac{f(x,s) - f(x,a_i)}{s-a_i}dx = \frac{d}{ds}\int_{\mathbb{R}} f(x,s)dx$$

The last equality holds by the definition of the derivative of $\int_{\mathbb{R}} f(x,s)dx$.

□

Problem E

$$h(x,y) := \frac{xy}{(x^2+y^2)^2}$$

(1) Explain why the integrals

$$\int_{\mathbb{R}} h(x,y)dx \text{ and } \int_{\mathbb{R}} h(x,y)dy$$

exist (in the Lebesgue sense) for any y or for any x, respectively, and evaluate these integrals.

(2) Explain why the integrals

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} h(x,y)dx\right)dy \text{ and } \int_{\mathbb{R}}\left(\int_{\mathbb{R}} h(x,y)dy\right)dx$$

exist (in the Lebesgue sense), and evaluate these integrals.

(3) Show that

$$\int_{\mathbb{R}^2} h(a)da,$$

where $a \in \mathbb{R}^2$ and da is the usual Lebesgue measure on $\mathbb{R}^2$, does not exist.

Solution:

(1) The integrals exist in the Lebesgue sense iff $|h(x,y)|$ is integrable with respect to x and y (respectively).

$$\begin{aligned}\int_{\mathbb{R}} |h(x,y)|dx &= \int_{-\infty}^{\infty}\left|\frac{xy}{(x^2+y^2)^2}\right|dx \\ &= 2\int_0^{\infty}\frac{x|y|}{(x^2+y^2)^2}dx \\ &= |y|\int_0^{\infty}\frac{2x}{(x^2+y^2)^2}dx \\ &= |y|\left(-(x^2+y^2)^{-1}\Big|_0^{\infty}\right) \\ &= \frac{|y|}{y^2} \\ &= \frac{1}{|y|} \\ &< \infty\end{aligned}$$

Which shows that the function is integrable in the Lebesgue sense.
To evaluate the integral:

$$\begin{aligned}\int_{\mathbb{R}} h(x,y)dx = \lim_{a\to\infty}\int_{-a}^{a}\frac{xy}{(x^2+y^2)^2}dx \\ &\text{This is an odd function, so the integral over } (-a,a) = 0 \text{ for any } a\text{:} \\ &= 0\end{aligned}$$

The same holds for $\int_{\mathbb{R}} h(x,y)dy$ with the change of variables $x = y$.

(2) In part (a), we evaluated the inner integrals to 0. Since 0 is Lebesgue integrable, these two double integrals evaluate to 0.

(3) Look at the double integral of $|h(a)|$. This is an even function, so we can change the domain to $[0,\infty)\times[0,\infty)$:

$$\begin{aligned}\int_{\mathbb{R}^2}|h(a)|da &= 4\int_{[0,\infty)^2}\left|\frac{xy}{(x^2+y^2)^2}\right|da \\ &\geq 4\int_0^{\infty}\int_0^{y}\left|\frac{xy}{(x^2+y^2)^2}\right|dxdy \\ &= 4\int_0^{\infty}\left(\frac{-y}{2(x^2+y^2)}\Big|_0^{y}\right)dy \\ &= 4\int_0^{\infty}\frac{1}{4y}dy \\ &= \int_0^{\infty}\frac{1}{y}dy \\ &= \infty, \text{ since the integral does not exist (compare with the harmonic series)}\end{aligned}$$

Which shows that the double integral does not exist.

□

Problem F

Show carefully (that is, with any necessary convergence arguments) that, if $\{\sigma_n : n \in \mathbb{Z}^+\}$ is an orthonormal basis for a Hilbert space H, with respect to an inner product $\langle \cdot, \cdot \rangle$ on H, then for any $f, g \in H$ we have

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, \sigma_n \rangle \langle \sigma_n, g \rangle$$

(meaning that the partial sums of the series on the right converge to the quantity on the left).

Solution:
First, we will show that if $\{\sigma_n\}$ is an ON basis for H, then for any $f \in H$:

$$f = \sum_{n=1}^{\infty} \langle f, \sigma_n \rangle \sigma_n$$

where the sum on the right has only countably many nonzero terms and converges in the norm topology no matter how these terms are ordered.
Let $\sigma_1, \sigma_2, ...$ be an enumeration of the σ_n's such that $\langle f, \sigma_n \rangle \neq 0$.
By Bessel's inequality, the series $\sum\limits_{n=1}^{\infty} |\langle f, \sigma_n \rangle|$ converges, so by the Pythagorean theorem:

$$\left\| \sum_{n=k}^{M} \langle f, \sigma_n \rangle \sigma_n \right\|^2 = \sum_{n=k}^{M} |\langle f, \sigma_n \rangle|^2 \to 0 \text{ as } k, M \to \infty$$

So the series converges to something in H, since H is complete.

$$\begin{aligned} f - \sum_{n=1}^{\infty} \langle f, \sigma_n \rangle \sigma_n \neq 0 &\Leftrightarrow \langle f - \sum_{n=1}^{\infty} \langle f, \sigma_n \rangle \sigma_n, \sigma_\alpha \rangle \neq 0 \forall \alpha \\ &\Leftrightarrow \langle f, \sigma_\alpha \rangle - \sum_{n=1}^{\infty} \langle f, \sigma_n \rangle \langle \sigma_n, \sigma_\alpha \rangle \neq 0 \text{ for some } \alpha \\ &\text{Since } \langle \sigma_n, \sigma_\alpha \rangle = 0 \text{ for all } n \neq \alpha : \\ &\Leftrightarrow \langle f, \sigma_\alpha \rangle - \langle f, \sigma_\alpha \rangle \neq 0 \text{ for some } \alpha \\ &\Leftrightarrow 0 \neq 0 \\ &\text{But this is a contradiction, so:} \end{aligned}$$

$$f - \sum_{n=1}^{\infty} \langle f, \sigma_n \rangle \sigma_n = 0 \Rightarrow f = \sum_{n=1}^{\infty} \langle f, \sigma_n \rangle \sigma_n$$

As desired.
Now, this fact can be applied to both $f, g \in H$:

$$
\begin{aligned}
\langle f, g\rangle &= \left\langle \sum_{n=1}^{\infty} \langle f, \sigma_n\rangle \sigma_n, \sum_{m=1}^{\infty} \langle g, \sigma_m\rangle \sigma_m \right\rangle \\
&= \sum_{n=1} \langle f, \sigma_n\rangle \sum_{m=1}^{\infty} \overline{\langle g, \sigma_m\rangle} \langle \sigma_n, \sigma_m\rangle \\
&\text{Using the property of inner product: } \overline{\langle x, y\rangle} = \langle y, x\rangle : \\
&= \sum_{n=1} \langle f, \sigma_n\rangle \sum_{m=1}^{\infty} \langle \sigma_m, g\rangle \langle \sigma_n, \sigma_m\rangle \\
&\text{Again, note: } \langle \sigma_n, \sigma_m\rangle = 0 \text{ for all } n \neq m : \\
&= \sum_{n=1}^{\infty} \langle f, \sigma_n\rangle \langle \sigma_n, g\rangle
\end{aligned}
$$

□