Analysis Prelim January 2016

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Problem 1

If (X, Σ, μ) is a measure space and if f is μ integrable, show that for every $\epsilon > 0$ there is $E \in \Sigma$ such that $\mu(E) < \infty$ and $\int_{X \setminus E} |f| d\mu < \epsilon$.

Solution 1:
Fix $\epsilon > 0$. Since f is integrable, the nonnegative function $|f|$ is integrable over E.
By definition of the integral of a nonnegative function, there is a bounded measurable function g on E which vanishes outside a subset E_0 of E with finite measure, and for which $0 \le g \le |f|$ and $\int_E (|f| - g) < \epsilon$. Then:

$$\int_{E \setminus E_0} |f| = \int_{E \setminus E_0} [|f| - g] \le \int_E [|f| - g] < \epsilon$$

$\square$

Solution 2: Define the measurable sets A_n for $n = 1, 2, 3, ...$:

$$A_n = \{x \in X : 1/n \le |f(x)| < n\}$$

$$A_0 = \{x \in X : f(x) = 0\} \text{ and } A_\infty = \{x \in X : f(x) = \infty\}$$

Each A_i is measurable, because it is the pre-image under f of a measurable subset of $\mathbb{R}$, and f is measurable so the preimages of measurable sets are measurable.
Define A:

$$A = \bigcup_{n=1}^{\infty} A_n$$

X is the union:

$$X = A_0 \cup A \cup A_\infty$$

Since f is integrable, we know $\int_X |f| d\mu < \infty$. Using the A_n's:

$$\int_X |f| d\mu = \int_{A_0} |f| d\mu + \int_A |f| d\mu + \int_{A_\infty} |f| d\mu < \infty$$

Since $f(x) = 0$ for all $x \in A_0$, we know $\int_{A_0} |f| d\mu = 0$.
Also, f is integrable, so the set on which $f(x) = \infty$ must have measure zero: $\mu(A_\infty) = 0$, so $\int_{A_\infty} |f| d\mu = 0$.
Then:

$$\int_X |f| d\mu = \int_A |f| d\mu = \lim_{n \to \infty} \int_{A_n} |f| d\mu < \infty$$

For any fixed $\epsilon > 0$, there necessarily exists A_N such that

$$\int_{X \setminus A_N} |f| d\mu < \epsilon$$

Since we have shown that the A_N are measurable, it remains only to show that the measure of A_N is finite:

$$\mu(A_N) \le N \int_{A_N} |f| d\mu \le N \int_X |f| d\mu < \infty$$

$\square$

Problem 2

Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$, and suppose that for every $a > 0$ the infinite series

$$\sum_{n=1}^{\infty} \mu(\{x \in [0,1] : |f_n(x)| > a\})$$

converges, where μ is the Lebesgue measure. Prove that

$$\lim_{n\to\infty} f_n(x) = 0$$

Solution:

Fix an arbitrary $x \in [0,1]$, and $\epsilon > 0$. We need to show that there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$|f_n(x)| < \epsilon$$

We know that, for any fixed $a > 0$:

$$\sum_{n=1}^{\infty} \mu(\{x \in [0,1] : |f_n(x)| > a\}) < \infty$$

Let $a = \epsilon/2$.The Borel-Cantelli Lemma tells us that almost all $x \in \mathbb{R}$ belong to at most finitely many of the sets $\{x \in [0,1] : |f_n(x)| > a\}$. Suppose E_0 is the subset of $\mathbb{R}$ of measure zero where $x \in E_0$ belong to infinitely many of the sets $\{x \in [0,1] : |f_n(x)| > a\}$, and consider $x \in \mathbb{R} \setminus E_0$.
x belongs to at most finitely many of the sets $\{x \in [0,1] : |f_n(x)| > a\}$, so there exists $N \in \mathbb{N}$ such that $x \notin \{x \in [0,1] : |f_n(x)| > a\}$ for any $n \geq N$.
This means $|f_n(x)| \leq a = \epsilon/2 < \epsilon$ for all $n \geq N$.
We can find such an N for all $x \in \mathbb{R} \setminus E_0$, and E_0 has measure zero, so $f_n(x) \to 0$ on $\mathbb{R} \setminus E_0$, which means

$$\lim_{n\to\infty} f_n(x) = 0$$

as desired.

$\square$

Problem 3

Let $f : [a,b] \to \mathbb{R}$ be a continuous function.

(a) Let $h > 0$. Show that the function

$$g_h(x) = \sup_{0<t<h} \frac{f(x+t) - f(x)}{t}$$

ismeasurable

(b) Show that $g(x) = \limsup_{t\to 0^+} \frac{f(x+t)-f(x)}{t}$ is measurable

(c) Prove that the set of points where f is differentiable is measurable.

Solution:

(a) Since $f(x)$ is measurable, $f(x+t) - f(x)$ is measurable for every t. Then, $1/t$ is measurable, as it is a constant function, so $\frac{f(x+t)-f(x)}{t}$ is measurable for all t.
The supremum of measurable functions is measurable as well, so $g_h(x)$ is measurable for every h.

(b) The limit of measurable functions is measurable, so $g(x)$ is measurable.

(c) As defined above, $g(x)$ is the upper derivative of f, say $\overline{D}(f) = g$.
By a symmetric argument, we can show that the lower derivative $\underline{D}(f)$ is also measurable.
f is differentiable at x iff $\overline{D}f(x) = \underline{D}f(x) < \infty$.
Equivalently, f is differentiable at x iff $\overline{D}(f)(x) - \underline{D}(f)(x) = 0$. Since both of these are measurable functions, their difference is also a measurable function. The inverse image of $\{0\}$ under this measurable function must be measurable in the domain, so the set of points where f is differentiable is measurable.

$\square$

Problem 4

Let f be integrable on the real line with respect to the Lebesgue measure. Evaluate

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x-n)\left(\frac{x}{1+|x|}\right) dx$$

Justify all steps.
Page 11 of Hawaii Real Analysis Prelim Solutions- Number 3

Solution:
First, use the substitution of variables $y = x - n$:

$$\int_{-\infty}^{\infty} f(x-n)\left(\frac{x}{1+|x|}\right) dx = \int_{-\infty}^{\infty} f(y)\left(\frac{y+n}{1+|y+n|}\right) dy$$

Next, split up the integral based on where $y + n > 0$ and where $y + n \leq 0$:

$$\int_{-\infty}^{\infty} f(y)\left(\frac{y+n}{1+|y+n|}\right) dy = \int_{-\infty}^{-n} f(y)\left(\frac{y+n}{1-(y+n)}\right) dy + \int_{-n}^{\infty} f(y)\left(\frac{y+n}{1+y+n}\right) dy$$

Now we can investigate the limit as $n \to \infty$ as it applies to each of these two integrals individually:
First Integral on the right:
Define the sequence of functions:

$$h_n(y) = f(y)\frac{y+n}{1-(y+n)}\chi_{(-\infty,-n]}$$

Note that $\lim\limits_{n\to\infty} h_n(y) = 0$.
For all n, and for $y \leq -n$: $\frac{|y+n|}{|1-(y+n)|} < 1$. It follows that

$$f(y)\frac{|y+n|}{|1-(y+n)|} < f(y)$$

f is integrable, so the Lebesgue dominated convergence theorem applies and we can justify passing the limit through the integral:

$$\lim_{n\to\infty} \int_{-\infty}^{-n} f(y)\left(\frac{y+n}{1-(y+n)}\right) dy = \lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(y)dy = \int_{-\infty}^{\infty} 0dy = 0$$

Second Integral on the right:
Define the sequence of functions for $y \in [-n, \infty)$:

$$g_n(y) = f(y)\frac{y+n}{1+y+n}\chi_{[-n,\infty)]}$$

Note that $g_n(y) \to f(y)$ as $n \to \infty$.
Also, this sequence is bounded above by $f(y)$, which is integrable, so the Lebesgue dominated convergence theorem applies and we can pass the limit through the integral here as well:

$$\lim_{n\to\infty} \int_{-n}^{\infty} f(y) \left(\frac{y+n}{1+y+n} \right) dy = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(y) dy = \int_{-\infty}^{\infty} f(y) dy$$

Putting both integrals together, we can return to the original question:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x-n) \left(\frac{x}{1+|x|} \right) dx = \int_{-\infty}^{\infty} f(y) dy$$

□

Problem 5

Let f be a nonnegative measurable function on $(-\infty, \infty)$ such that $f(x) < \infty$ μ-almost everywhere; here μ is the Lebesgue measure. Prove or give a counterexample to each of the following:

(a) For every $N \in \mathbb{N}$ there exists a compact K such that $\mu(K) > N$ and f is integrable over K.

(b) There exist $a < b$ such that f is integrable over $[a, b]$.

Solution:

(a) This is Lusin's Theorem.
Fix $N \in \mathbb{N}$.
Since f is measurable on $(-\infty, \infty)$, f is measurable on $[N, N]$. Pick $\epsilon > 0$ such that $\epsilon < N$. By Lusin's Theorem, there exists a compact set $K \subset [-N, N]$ such that $m([-N, N] \setminus K) < \epsilon$ and f is continuous on K.
Since $m([-N, N] \setminus K) = 2N - \mu(K) < \epsilon < N$, it follows that $\mu(K) > N$.

(b) This follows from the fact that $\mu(K) > 0$. Since K is compact, it cannot have nonempty interior, so it follows that K contains some closed interval $[a, b]$.

□

Problem 6

A C^∞ function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition for each $x \in \mathbb{R}$ there exists $n_x \in \mathbb{N}$ such that $f^{(n_x)}(x) = 0$. Prove that f is a polynomial.

Solution:
Suppose f is not a polynomial everywhere on $\mathbb{R}$.
Define the following closed sets:

$$S_n = \{x : f^{(n)}(x) = 0\}$$

The sets are closed because any Cauchy sequence in any S_n must converge to another element of S_n.
Also, define the set X:

$$X = \{x : \forall (a, b) \text{ containing } x, f|_{(a,b)} \text{ is not a polynomial}\}$$

X is nonempty, because we are assuming f is not a polynomial
X is also closed, and it contains no isolated points, because of how the intervals (a, b) are used to describe the elements of X.
X is also complete, so by the Baire category theorem X is not the countable union of nowhere dense sets.

$$X = \bigcup_{n=0}^{\infty} (X \cap S_n)$$

So there exists an interval (a, b) such that $(a, b) \cap X \subset S_n$ for some n.
Every $x \in (a, b) \cap X$ is in S_n, so $f^{(n)}(x) = 0$ and $f(x)$ is not expressible as a polynomial.
$x \in (a, b) \cap X$, and there is an open interval around x (namely (a, b)) on which $f^{(n)} = 0$. When taking the next derivative at x, we can treat the function $f^{(n)}$ as the constant zero function around x, so clearly $f^{(n+1)}(x) = 0$. This works for all following derivatives, so $x \in S_m$ for all $m \geq n$ and $x \in (a, b) \cap X$.
This means that the polynomial f *can* be approximated by an at most $(n-1)$th degree polynomial on $(a, b) \cap X$, because the nth derivative and all further derivatives are zero, so we can use a Taylor series to approximate f on $(a, b) \cap X$.
We will show that we can also use a polynomial of degree $< n$ to approximate f on the rest of (a, b). This will give us a polynomial approximation on any open interval (a, b), and these approximations can be glued together at their intersections to give us a polynomial expression of f on all of $\mathbb{R}$.
Now, take a maximal open interval $(c, e) \subset ((a, b) \setminus X)$. (a, b) is bounded and X is closed, so we know such an interval exists. f has a polynomial expression on (c, e) of some degree, say d. Then $f^{(d)} =$ some constant $\neq 0$ on (c, e). Since the value of the derivative is defined via limiting process, we know $f^{(d)} \neq 0$ on $[c, e]$. Thus, $d < n$, because $f^{(n)}(c) = f^{(n)}(e) = 0$ and all derivatives greater than n are zero at c and e, since $c, e \in X$.
This means we can express f as a polynomial of fixed degree everywhere on (a, b), and by the discussion above f is a polynomial everywhere on $\mathbb{R}$.

$\square$