Analysis Prelim August 2016

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Problem 1

Let m be Lebesgue measure on $\mathbb{R}$, and let $E \subset \mathbb{R}$ have finite Lebesgue measure. If $E_r := \{x \in E : |x| > r\}$, prove that $m(E_r) \to 0$ as $r \to \infty$.

Solution:
If $\{r_n\}_{n=1}^{\infty}$ is an increasing sequence such that $r_n \to \infty$ as $n \to \infty$, then $E_{r_n} \supseteq E_{r_{n+1}}$ for all $n = 1, 2, ...$ So $\{E_{r_n}\}$ is a decreasing sequence of sets, and we know $\lim\limits_{n\to\infty} m(E_{r_n})$ must exist and be finite, since $m(E) < \infty$.
Also $\bigcap\limits_{n=1}^{\infty} E_{r_n} = \varnothing$, by construction.
Define the sets F_n:

$$F_n := E \setminus E_n$$

Notice that $F_1 \subset F_2 \subset \cdots$, so $F_n \nearrow E$.
By continuity of measure, $m(F_n) \nearrow m(E)$.
By the excision property, $m(F_n) = m(E) - m(E_n)$.
Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} m(F_n) = m(E) - \lim_{n\to\infty} m(E_{r_n}) = m(E)$$

So we conclude that $\lim\limits_{n\to\infty} m(E_{r_n}) = 0$

$\square$

Problem 2

Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of measurable functions. Suppose that

(i) $\int_0^1 |f_n|^2 \leq 1$ for $n = 1, 2, ...$, and

(ii) $f_n \to 0$ almost everywhere.

Show that

$$\lim_{n\to\infty} \int_0^1 f_n = 0$$

Solution:

Fix a number $M > 0$ and split up the desired integral:

$$\int_0^1 f_n = \underbrace{\int_0^1 f_n \chi_{\{x:f(x)>M\}}}_{I} + \underbrace{\int_0^1 f_n \chi_{\{x:f(x)\leq M\}}}_{II} \tag{1}$$

Looking at these two integrals separately:
Integral I:

$$\begin{aligned}\int_0^1 f_n \chi_{\{x:f(x)>M\}} &\leq \int_0^1 f_n \cdot f_n \cdot \frac{1}{M} \\ &= \frac{1}{M}\int_0^1 (f_n)^2 \chi_{\{x:f(x)>M\}} \\ &\text{And since } (f_n)^2 \text{ is nonnegative and } \chi_{\{x:f(x)>M\}} \leq 1 \text{ on } [0,1]\text{:} \\ &\leq \frac{1}{M}\int_0^1 (f_n)^2 \\ &\text{And using hypothesis (i):} \\ &\leq \frac{1}{M}\end{aligned} \tag{2}$$

Integral II:
The function $f_n \chi_{\{x:f(x)\leq M\}}$ is dominated by the integrable function M on $[0,1]$, so we can apply the Lebesgue dominated convergence theorem:

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 f_n \chi_{\{x:f(x)\leq M\}} &= \int_0^1 \lim_{n\to\infty} f_n \chi_{\{x:f(x)\leq M\}} \\ &= 0\end{aligned} \tag{3}$$

Plugging the information from (2) and (3) into (1):

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 f_n &= \lim_{n\to\infty}\int_0^1 f_n \chi_{\{x:f(x)>M\}} + \lim_{n\to\infty}\int_0^1 f_n \chi_{\{x:f(x)\leq M\}} \\ &= \frac{1}{M}\end{aligned}$$

And $1/M$ can be made arbitrarily small by choosing a large enough M, so the desired result holds.

$\square$

Problem 3

Let f and g be real-valued integrable functions on a measure space $(X, \mathcal{B}, \mu)$, and define

$$F_t = \{x \in X : f(x) > t\},\ G_t = \{x \in X : g(x) > t\}$$

Prove that

$$\int |f - g| d\mu = \int_{-\infty}^{\infty} \mu((F_t \setminus G_t) \cup (G_t \setminus F_t)) dt$$

Hint: Rewrite the right-hand side as a double integral.

Solution:

Starting with the suggested hint:

$$\begin{aligned}
\int_{-\infty}^{\infty} \mu((F_t \setminus G_t) \cup (G_t \setminus F_t))dt &= \int_{-\infty}^{\infty} \int_{(F_t \setminus G_t)\cup(G_t \setminus F_t)} 1dxdt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(F_t \setminus G_t)\cup(G_t \setminus F_t)}(x)dxdt \\
&\text{Since the sets } F_t \setminus G_t \text{ and } G_t \setminus F_t \text{ are disjoint:} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\chi_{(F_t \setminus G_t)}(x) + \chi_{(G_t \setminus F_t)}(x))dxdt \\
&\text{We will justify switching the order of the integrals by Fubini-Tonelli Theorem:} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(F_t \setminus G_t)}(x)dtdx + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(G_t \setminus F_t)}(x)dtdx
\end{aligned} \tag{4}$$

Considering the inner integral $\int_{-\infty}^{\infty} \chi_{(F_t \setminus G_t)}(x)dt$, note that the x value is fixed since we switched the order of integration. We are integrating over all values of t such that $g(t) \leq t < f(t)$, so we are really looking at the size of the image set where $f(x) > g(x)$. This set is size $f(x) - g(x)$, for each x where $f(x) > g(x)$. This gives us a valuation for the integral:

$$\int_{-\infty}^{\infty} \chi_{(F_t \setminus G_t)}(x)dt = (f(x) - g(x))\chi_{\{x:f(x)>g(x)\}}$$

We likewise get a definition for the inner integral in the second term:

$$\int_{-\infty}^{\infty} \chi_{(G_t \setminus F_t)}(x)dt = (g(x) - f(x))\chi_{\{x:f(x)<g(x)\}}$$

Putting this information back into (4):

$$\begin{aligned}
\int_{-\infty}^{\infty} \mu((F_t \setminus G_t) \cup (G_t \setminus F_t))dt &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(F_t \setminus G_t)}(x)dtdx + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(G_t \setminus F_t)}(x)dtdx \\
&= \int_{-\infty}^{\infty} ((f(x) - g(x))\chi_{\{x:f(x)<g(x)\}} + (g(x) - f(x))\chi_{\{x:f(x)<g(x)\}})dx \\
&\text{If } f(x) = g(x)\text{, then } f(x) - g(x) = 0\text{, so we can add this zero term:} \\
&= \int_{-\infty}^{\infty} ((f(x) - g(x))\chi_{\{x:f(x)<g(x)\}} + (g(x) - f(x))\chi_{\{x:f(x)<g(x)\}} + (f(x) - g(x))\chi_{\{x:f(x)=g(x)\}} \\
&= \int_{-\infty}^{\infty} |f(x) - g(x)|dx
\end{aligned}$$

□

Problem 4

Let $f \in L^1(\mathbb{R})$ be a function satisfying $\int_{\mathbb{R}} |f(x)|dx = 1$.

(a) Prove that

$$\lim_{|t|\to\infty} \int_{\mathbb{R}} f(x)\cos(tx)dx = 0$$

Justify your reasoning.

(b) Compute

$$\lim_{t\to+\infty} \int_{\mathbb{R}} |f(x)\sin^2(tx)|dx$$

Solution:

(a) Simple functions are dense in $L^1(\mathbb{R})$ with respect to the L^1-norm.
Suppose first that $f(x)$ is a simple function:

$$f(x) := \sum_{i=1}^{N} c_i \chi_{(a_i, b_i)}$$

Consider the integral in question:

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(x) \cos(tx) dx \right| &= \left| \sum_{i=1}^{N} \int_{a_i}^{b_i} c_i \cos(tx) dx \right| \\
&\leq \sum_{i=1}^{N} \frac{|c_i|}{|t|} |\sin(tb_i) - \sin(ta_i)| \\
&\leq \sum_{i=1}^{N} \frac{2|c_i|}{|t|} \\
&\text{Taking the limit:} \\
\lim_{|t| \to \infty} \left| \int_{\mathbb{R}} f(x) \cos(tx) dx \right| &\leq \lim_{|t| \to \infty} \sum_{i=1}^{N} \frac{2|c_i|}{|t|} \\
&= 0
\end{aligned}$$

*This is a special version of the Riemann Lebesgue Lemma.

(b)

$$\begin{aligned}
\lim_{t \to +\infty} \int_{\mathbb{R}} |f(x) \sin^2(tx)| dx &= \lim_{t \to +\infty} \int_{\mathbb{R}} |f(x)| \sin^2(tx) dx \\
&= \lim_{\frac{1}{2}t \to +\infty} \int_{\mathbb{R}} |f(x) \sin^2(\frac{1}{2}tx)| dx \\
&\text{Using the double angle cosine trig identity:} \\
&= \lim_{\frac{1}{2}t \to +\infty} \int_{\mathbb{R}} |f(x)| \frac{1 - \cos(xt)}{2} dx \\
&= \lim_{t \to +\infty} \int_{\mathbb{R}} |f(x)| \frac{1 - \cos(xt)}{2} dx \\
&= \lim_{t \to +\infty} \frac{1}{2} \int_{\mathbb{R}} |f(x)| dx - \lim_{t \to +\infty} \frac{1}{2} \int_{\mathbb{R}} |f(x)| \cos(tx) dx \\
&= \frac{1}{2}
\end{aligned}$$

□

Problem 5

(a) Let $f : [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ be in $L^s([0,1])$, where $s \in (1, \infty)$. Suppose that $r \in [1, \infty)$ and $r < s$. Prove that $f \in L^r([0,1])$.

(b) Prove that $L^6(\mathbb{R}) \cap L^3(\mathbb{R}) \subset L^4(\mathbb{R})$, and moreover show that this containment is proper. Explain your reasoning.

Solution:

(a) There exists $t \in (0,1)$ such that $r = ts$.
Note that the function $x \mapsto a^x$ is a convex function, so by $Jensen'sinequality$:

$$|f|^r = |f|^{0(1-t)}|f|^{ts} \leq (1-t)|f|^0 + t|f|^s$$

So $|f|^r \leq (1-t) \cdot 1 + t|f|^s$. Integrating, by monotonicity of integration:

$$\begin{aligned}\int_0^1 |f|^r &\leq (1-t)\int_0^1 1 d\mu + t\int_0^1 |f|^s d\mu \\ &= \mu([0,1])(1-t) + t\,\|f\|_s^s\end{aligned}$$

Since $\mu([0,1]) = 1 < \infty$ and $f \in L^s([0,1])$, the righthand side is finite. Thus, $f \in L^r([0,1])$.

(b) There exists $t \in (0,1)$ such that:

$$4 = 6t + (1-t)3$$

In particular, $t = 1/3$.
Suppose $f \in L^6([0,1]) \cap L^3([0,1])$.
Note that the function $x \mapsto a^x$ is a convex function, so by Jensen's inequality:

$$|f|^4 = |f|^{6t+(1-t)3} \leq t|f|^6 + (1-t)|f|^3$$

By the monotonicity of integration:

$$\int_0^1 |f|^4 \leq t\int_0^1 |f|^6 + (1-t)\int_0^1 |f|^3$$

The righthand side is less than infinity because $t \in (0,1)$ and $f \in L^6([0,1]) \cap L^3([0,1])$.
Thus, $\int_0^1 |f|^4 < \infty$, so $f \in L^4([0,1])$.

This can be generalized: For any $1 \leq p < q < r < \infty$, if $f \in L^p \cap L^r$, then $f \in L^q$:
There exists $t \in (0,1)$ such that $q = tp + (1-t)r$.
$x \mapsto a^x$ is a convex function, so we can apply Jensen's inequality:

$$|f|^q = |f|^{tp+(1-t)r} \leq t|f|^p + (1-t)|f|^r$$

And integrate to get the desired result.

□

Problem 6

Let $C([0,1])$ be the Banach space of all complex-valued continuous functions on $[0,1]$ with norm

$$\|f\| = \sup_{x\in[0,1]} |f(x)|$$

(a) If we define $\mathbf{B}$ by

$$\mathbf{B} = \{f \in C([0,1]) : \|f\| \leq 1\}$$

show that $\mathbf{B}$ is a closed subset of $C([0,1])$ that is not compact.

(b) Let $H : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous function, and for $f \in C([0,1])$ define

$$S(f)(x) = \int_0^1 H(x,y)f(y)dy$$

Prove that if $f \in C([0,1])$ then $S(f) \in C([0,1])$, and also prove that the closure of $\{S(f) : f \in \mathbf{B}\}$ is compact in $C([0,1])$.

Solution:

(a) In compact metric spaces, the Bolzano-Weierstrass property holds: Every sequence has a convergent subsequence.
We will show that **B** is closed, but that the Bolzano-Weierstrass property does not hold, so **B** is not compact.
Closed:
Let $\{f_n\}$ be a Cauchy sequence of functions in **B**. Since $C([0,1])$ is complete, it must converge (in norm) to a function $f \in C([0,1])$:

$$\lim_{n\to\infty} \|f_n\| = \|f\|$$

Since $\{f_n\} \subset \mathbf{B}$, $\|f_n\| \leq 1$ for all n. The norm is a continuous function, so it follows that $\|f\| \leq 1$ as well. Thus, $f \in \mathbf{B}$, so **B** is closed.
To show that **B** is not compact, we will show that not every sequence has a convergent subsequence.
Pointwise limit of continuous functions is not necessarily continuous: Consider the continuous functions $\{f_n(x) := x^n\} \subset \mathbf{B}$. The pointwise limit of this sequence is:

$$f := \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Which is not a continuous function, so it cannot be in **B**. Thus, the Bolzano-Weierstrass property does not hold, so **B** is not compact.

(b) Define:

$$\mathbf{A} := \overline{\{S(f) : f \in \mathbf{B}\}}$$

First, we must show $S(f) \in C([0,1])$. Fix $\epsilon > 0$.
Notice that H is continuous on a compact set, so H is uniformly continuous. In particular, H is uniformly continuous in the first variable: There exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$ then $|H(x_1, y) - H(x_2, y)| < \epsilon$, for all $y \in [0,1]$.
Choose x_1, x_2 to satisfy this condition:

$$\begin{aligned} |S(f)(x_1) - S(f)(x_2)| &= \left| \int_0^1 f(y)(H(x_1,y) - H(x_2,y)) \right| \\ &\leq \|f\| \int_0^1 |H(x_1,y) - H(x_2,y)| dy \\ &\text{By continuity of } H \text{ and the fact that } \|f\| \leq 1\text{:} \\ &< \int_0^1 \epsilon dy \\ &= \epsilon \end{aligned}$$

So $S(f) \in C([0,1])$.
By Arzela-Ascoli, **A** is compact if and only if it is closed, bounded and equicontinuous.
By definition of closure, **A** is closed.
The elements of **B** are bounded and norm is continuous, so **A** is bounded as well.
It remains to show that **A** is equicontinuous. This follows from the work above, as the continuity of $S(f)$ did not depend on the choice of f.

$\square$