Analysis Prelim August 2014

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Problem 1

Prove the following statement or provide a counterexample to the following statement: There exists an open subset, E, of the closed interval on the real line, $[0,1]$, with the following two properties:

(a) Lebesgue measure of $\{E \cap (a,b)\} > 0$ for all non-empty open subintervals, (a,b), of $[0,1]$ with $0 < a < b < 1$.

(b) Lebesgue measure of $E < 1$.

Solution:
Let $\{r_n\}$ enumerate the rationals in $[0,1]$.
Let V_1 be a segment of finite length centered at r_1, and define V_n to be a segment of length $m(V_{n-1})/3$ centered at r_n for all $n = 2, 3, ...$
Also define W_n:

$$W_n := V_n - \bigcup_{k=1}^{\infty} V_{n+k}$$

By construction, the W_n are disjoint from each other.
Every open interval contains infinitely many rational numbers, so it contains a W_n for some n.
Looking at the measure of W_n:

$$\begin{aligned} m(W_n) &\geq m(V_n) - \sum_{k=1}^{\infty} m(V_{n+k}) \text{ (by countable additivity)} \\ &= m(V_n) - m(V_n) \sum_{k=1}^{\infty} 3^{-k} \\ &= \frac{m(V_n)}{2} \\ &> 0 \end{aligned}$$

$m(W_n) > 0$, because the V_n were defined to all have positive measure.
Every set of positive measure strictly contains a closed set of positive measure, say:

$$A_n \subset W_n \text{ with } 0 < m(A_n) < m(W_n)$$

Define $A = \bigcup_{k=1}^{\infty} A_k$. Since $A_m \subset W_m$ and $A_n \subset W_n$ and $W_m \cap W_n = \varnothing$, the A_k are disjoint as well.
In terms of measure:

$$0 < m(A \cap W_n) = m(A_n) < m(W_n)$$

Every open interval contains a W_n, so every open interval intersects A nontrivially.
A does not have measure 1, because of how the V_n's were constructed.

□

Cleaner Solution, Maybe?:
The Smith-Volterra-Cantor set is nowhere dense (contains no intervals), but it has positive measure. The complement of this set in $[0,1]$ is dense (contains all intervals), and has measure less than 1.

To construct the Smith-Volterra-Cantor set, start with $[0,1]$.
First, remove the middle 1/4.
From the remaining two intervals, remove the middle 1/16. A total of 1/8 is removed.
From the remaining four intervals, remove the middle 1/64. A total of 1/16 is removed.
...On the nth step, a total of $2^{-(n+1)}$ is removed. Continue the process infinitely many times, and let S denote the Smith-Volterra-Cantor set.
By construction, this set contains no intervals (because we always remove the middle), so every open interval (a,b) is contained in the complement.
The measure of S is the measure of $[0,1]$ without the portions removed. By the countable additivity of measure and the fact that these removed portions are all of course disjoint:

$$m(S) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^{-(n+1)}} = \frac{1}{2}$$

Thus, $m(S^c) = \frac{1}{2}$ and $m(\{S^c \cap (a,b)\}) = b - a$ for all $(a,b) \subseteq [0,1]$.

$\square$

Problem 2

Let $f \in L^p(-\infty, \infty)$ where $1 \leq p \leq \infty$. Show that the function

$$F(t) = \int_0^t f(s)ds$$

is well defined and continuous.

Solution:
Begin by considering the different cases for the values of p:

- $p = 1$:
  If $f \in L^1(-\infty,\infty)$, then $\int_{\mathbb{R}} |f| < \infty$. This means that $|f|$ is finite almost everywhere, so possibly excising a set of measure 0, assume that f is finite on $(-\infty,\infty)$. There exists a finite $M > 0$ such that $|f(x)| \leq M$.
  Now, suppose $y > x$. We will show that F is continuous. Fix $\epsilon > 0$:
  $$\begin{aligned} |F(y) - F(x)| &= \left| \int_x^y f(s)ds \right| \\ &\leq \int_x^y |f(s)|ds \\ &\leq \int_x^y M ds \\ &= |y - x| M \end{aligned}$$
  If we choose x, y such that $|y - x| < \epsilon/M$, then $|F(y) - F(x)| < \epsilon$. F is continuous.

- $1 < p < \infty$:
  Note that $1/p + (p-1)/p = 1$, so $p/(p-1)$ is the conjugate of p.
  To show that F is continuous, pick $y > x$, two real numbers.
  $$\begin{aligned} |F(y) - F(x)| &= \left| \int_x^y f(s)ds \right| \\ &\leq \int_{\mathbb{R}} |\chi_{[x,y]}(s) \cdot f(s)|ds \\ &\leq \left\|\chi_{[x,y]}\right\|_{p/(p-1)} \cdot \|f\|_p \\ &= (\mu([x,y]))^{(p-1)/p} \cdot \|f\|_p \\ &= |y - x|^{(p-1)/p} \|f\|_p \end{aligned}$$

If we choose x, y such that $|y-x|^{(p-1)/p} < \epsilon / \|f\|_p$, then $|F(y) - F(x)| < \epsilon$. F is continuous.

- $p = \infty$:
  If $f \in L^\infty(-\infty, \infty)$, then we know $\|f\|_\infty < \infty$.
  Suppose $x < y$, without loss of generality.

$$\begin{aligned}
|F(y) - F(x)| &= \left| \int_x^y f(s)ds \right| \\
&= \left| \int_{\mathbb{R}} \chi_{[x,y]}(s) f(s) ds \right| \\
&\text{By Hölder's Inequality:} \\
&\leq \left\|\chi_{[x,y]}\right\|_1 \cdot \|f\|_\infty \\
&= \mu([x,y]) \|f\|_\infty \\
&= |y-x| \|f\|_\infty
\end{aligned}$$

So if we choose x and y such that $|y-x| < \epsilon / \|f\|_\infty$, then $|F(x) - F(y)| < \epsilon$.
This shows that F is continuous.

In all of these cases, F is well-defined, because if $t = t' \in \mathbb{R}$, then:

$$F(t) = \int_0^t f(s)ds = \int_0^{t'} f(s)ds = F(t')$$

□

Problem 3

Let $H_k(t), k = 0, 1, 2, \ldots$ be a sequence of functions on $[0,1]$ defined as follows: $H_0(t) \equiv 1$ and, if $2^n \leq k < 2^{n+1}$ where n is a nonnegative integer, then

$$H_k(t) = \begin{cases} 2^{n/2} & \text{if } \frac{k-2^n}{2^n} \leq t < \frac{k-2^n+0.5}{2^n} \\ -2^{n/2} & \text{if } \frac{k-2^n+0.5}{2^n} \leq t < \frac{k-2^n+1}{2^n} \\ 0 & \text{otherwise} \end{cases}$$

Show that, for every function f in the Hilbert space $L^2([0,1])$

$$\lim_{k \to \infty} \int_0^1 f(t) H_k(t) dt = 0$$

Solution:

The sizes of the intervals $[\frac{k-2^n}{2^n}, \frac{k-2^n+0.5}{2^n}]$ and $[\frac{k-2^n+0.5}{2^n}, \frac{k-2^n+1}{2^n}]$ are the same. This size is:

$$\frac{k-2^n+0.5}{2^n} - \frac{k-2^n}{2^n} = \frac{1}{2^{n+1}}$$

Consider the integral for a fixed k:

$$\begin{aligned}
\int_0^1 f(t) H_k(t) dt &= \int_0^{\frac{k-2^n}{2^{n/2}}} 0 \cdot f(t) dt + \int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+0.5}{2^n}} (2^{n/2}) f(t) dt + \int_{\frac{k-2^n+0.5}{2^n}}^{\frac{k-2^n+1}{2^n}} (-2^{n/2}) f(t) dt + \int_{\frac{k-2^n+1}{2^n}}^1 0 \cdot f(t) dt \\
&= \int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+0.5}{2^n}} (2^{n/2}) f(t) dt + \int_{\frac{k-2^n+0.5}{2^n}}^{\frac{k-2^n+1}{2^n}} (-2^{n/2}) f(t) dt
\end{aligned}$$

The continuous functions are dense in L^2, so start by supposing that f is continuous. We will show that the desired limit goes to 0, then we will show that it goes to 0 in general for L^2 functions.
If $f \in C([0,1])$, then f achieves a maximum value on $C([0,1])$. Suppose $M = \max_{x\in[0,1]} \{|f(x)|\}$.
Return to the equation above, considering the absolute value.

$$\begin{aligned}
\left|\int_0^1 f(t)H_k(t)dt\right| &= \left|\int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+0.5}{2^n}} (2^{n/2})f(t)dt + \int_{\frac{k-2^n+0.5}{2^n}}^{\frac{k-2^n+1}{2^n}} (-2^{n/2})f(t)dt\right| \\
&\leq \int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+0.5}{2^n}} (2^{n/2})|f(t)|dt + \int_{\frac{k-2^n+0.5}{2^n}}^{\frac{k-2^n+1}{2^n}} (2^{n/2})|f(t)|dt \\
&\leq 2^{n/2}M \int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+0.5}{2^n}} 1dt + 2^{n/2}M \int_{\frac{k-2^n+0.5}{2^n}}^{\frac{k-2^n+1}{2^n}} 1dt \\
&\text{By definition of Lebesgue Measure:} \\
&= 2^{n/2}M\frac{1}{2^{n+1}} + 2^{n/2}M\frac{1}{2^{n+1}} \\
&= 2^{-(n/2)-1}M + 2^{-(n/2)-1}M \\
&= 2^{-n/2}M
\end{aligned}$$

Taking the limit as $k \to \infty$ is equivalent to taking the limit as $n \to \infty$, since $2^n \leq k < 2^{n+1}$. Thus:

$$\lim_{k\to\infty} \int_0^1 f(t)H_k(t)dt = 0$$

Now, if g is any function in $L^2([0,1])$, g can be approximated arbitrarily closely by a continuous function. If we fix $\epsilon > 0$, then there exists $f \in C([0,1])$ such that

$$\|f - g\|_{L^2([0,1])} < \epsilon$$

Suppose f and g are as described above. We will show

$$\lim_{k\to\infty} \int_0^1 g(t)H_k(t)dt < \epsilon 0.$$

To do this, we will need to know the L^2-norm of H_k:

$$\begin{aligned}
\|H_k\|_2 &= \left(\int_0^1 |H_k(t)|^2 dt\right)^{1/2} \\
&= \left(\int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+1}{2^n}} 2^n dt\right)^{1/2} \\
&= \left(2^n\left(\frac{k-2^n}{2^n} - \frac{k-2^n+1}{2^n}\right)\right)^{1/2} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

Now, begin by expanding the desired integral:

$$\begin{aligned}
\int_0^1 g(t)H_k(t)dt &= \int_0^1 (g(t) - f(t))H_k(t)dt + \int_0^1 f(t)H_k(t)dt \\
&\text{By Cauchy-Schwarz:} \\
&\leq \|f - g\|_2 \cdot \|H_k\|_2 + \int_0^1 f(t)H_k(t)dt \\
&< \epsilon \cdot \frac{1}{\sqrt{2}} + \int_0^1 f(t)H_k(t)dt
\end{aligned}$$

Taking the limit as $k \to \infty$, we know that the integral on the right goes to 0, so:

$$\lim_{k\to\infty} \int_0^1 g(t)H_k(t)dt < \frac{\epsilon}{\sqrt{2}}$$

Which establishes the desired result.

$\square$

Problem 4

Consider the expression

$$\int_0^\infty \frac{\sin(x)}{x^\alpha} dx$$

Does there exist an $\alpha > 0$ such that the given integral expression exists as an improper Riemann integral but does not exist as a Lebesgue integral? Prove your answer.

Solution:
Let $\alpha = 1$. Note that $\int_0^\infty e^{-xy}\sin(x)dy = \frac{\sin(x)}{x}$. Investigating the integral of this expression:

$$\begin{aligned}\int_0^\infty \frac{\sin(x)}{x} dx &= \int_0^\infty \left(\int_0^\infty e^{-xy}\sin(x)dy\right) dx \\ &\text{We will justify switching the order of integration by Fubini's.} \qquad (1) \\ &= \int_0^\infty \left(\int_0^\infty e^{-xy}\sin(x)dx\right) dy\end{aligned}$$

Evaluate the inner integral of (1) using integration by parts:

$$\begin{aligned}\int_0^\infty e^{-xy}\sin(x)dx &= -e^{-xy}\cos(x)\Big|_0^\infty - y\int_0^\infty e^{-xy}\cos(x)dx \\ &= 1 - y\int_0^\infty e^{-xy}\cos(x)dx \\ &\text{Using integration by parts a second time:} \\ &= 1 - y\left(\sin(x)e^{-xy}\Big|_0^\infty + y\int_0^\infty e^{-xy}\sin(x)dx\right) \\ &= 1 - y\left(0 + y\int_0^\infty e^{-xy}\sin(x)dx\right) \qquad (2) \\ \int_0^\infty e^{-xy}\sin(x)dx &= 1 - y^2\left(\int_0^\infty e^{-xy}\sin(x)dx\right) \\ &\text{Combining like terms:} \\ (1+y^2)\int_0^\infty e^{-xy}\sin(x)dx &= 1 \\ &\text{Solving:} \\ \int_0^\infty e^{-xy}\sin(x)dx &= \frac{1}{1+y^2}\end{aligned}$$

Now we can plug the information from (2) into (1):

$$\begin{aligned}\int_0^\infty \frac{\sin(x)}{x} dx &= \int_0^\infty \left(\int_0^\infty e^{-xy}\sin(x)dx\right) dy \\ &= \int_0^\infty \frac{1}{1+y^2} dy \\ &= \frac{\pi}{2}\end{aligned}$$

So the function is Riemann integrable.
However, the function is not absolutely integrable, so it is *not* Lebesgue integrable:

$$\int_0^\infty \left|\frac{\sin(x)}{x}\right| dx = \lim_{N\to\infty}\sum_{k=0}^{N}\int_{k\pi}^{(k+1)\pi}\left|\frac{\sin(x)}{x}\right| dx$$

$$\text{Use the change of variables } x \mapsto t + k\pi :$$

$$\begin{aligned}
&= \lim_{N\to\infty}\sum_{k=0}^{N}\int_{k\pi}^{(k+1)\pi}\left|\frac{\sin(x)}{x}\right| dx \\
&= \lim_{N\to\infty}\sum_{k=0}^{N}\int_{0}^{\pi}\left|\frac{\sin(t+k\pi)}{t+k\pi}\right| dt \\
&\geq \lim_{N\to\infty}\sum_{k=0}^{N}\frac{1}{\pi+k\pi}\int_0^\pi |\sin(t+k\pi)|\, dt \\
&\geq \lim_{N\to\infty}\sum_{k=0}^{N}\frac{1}{\pi+k\pi} \\
&= \sum_{k=0}^{\infty}\frac{1}{\pi+k\pi} \\
&\to \infty
\end{aligned}$$

The last line diverges since the harmonic series diverges.

□

Problem 5

Let A_k be a sequence of measurable subsets of $[0,1]$ such that, for every finite set of indices $i_1 < i_2 < \cdots < i_k$,

$$m(A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}) = m(A_{i_1})m(A_{i_2})\cdots m(A_{i_k})$$

where m stands for the Lebesgue measure.

(a) Show that the sequence $B_k = [0,1]\setminus A_k$ has the same property. (*Hint:* Show that, if the property holds for the sequence A_k, then it still holds if exactly one of the sets A_k is replaced by the corresponding B_k).

(b) Suppose in addition that the series $\sum m(A_k)$ diverges. Show that

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = 1$$

Solution:

(a) (Typed up by Leo):
Observe that $[0,1] = A_{i_j}\bigsqcup B_{i_j}$ by definition. Intersections distribute, so $A_{i_1}\cap\cdots\cap\widehat{A_{i_j}}\cap\cdots\cap A_{i_k} = (A_{i_1}\cap\cdots\cap A_{i_k})\bigsqcup(A_{i_1}\cap\cdots\cap B_{i_j}\cap\cdots\cap A_{i_k})$. Measures turn $\bigsqcup$ into sums. The symbol $\widehat{\cdot}$ means omit $\cdot$.

$$m(A_{i_1}\cap\cdots\cap\widehat{A_{i_j}}\cap\cdots\cap A_{i_k}) = m(A_{i_1}\cap\cdots\cap A_{i_k}) + m(A_{i_1}\cap\cdots\cap B_{i_j}\cap\cdots\cap A_{i_k})$$

Now for prong #2: subtract.

$$
\begin{aligned}
m(A_{i_1} \cap \cdots \cap \widehat{A_{i_j}} \cap \cdots \cap A_{i_k}) - m(A_{i_1} \cap \cdots \cap A_{i_k}) &= m(A_{i_1}) \cdots \widehat{m(A_{i_j})} \cdots m(A_{i_k}) - m(A_{i_1}) \cdots m(A_{i_k}) \\
&= m(A_{i_1}) \cdots \widehat{m(A_{i_j})} \cdots m(A_{i_k})(1 - m(A_{i_j})) \\
&= m(A_{i_1}) \cdots m(B_{i_j}) \cdots m(A_{i_k})
\end{aligned}
$$

We have shown the property for only a single A_{i_j} replaced by B_{i_j}; it remains to show it for all. Induct on k and follow the above argument. This will lead you to your goal, for any fixed collection of i_j's, and therefore for all finite such.

(b) Note that showing $m\left(\bigcup_{k=1}^{\infty} A_k\right) = 1$ is equivalent to showing $m\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right) = 0$. Looking at this measure:

$$
\begin{aligned}
m\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right) &= m\left(\bigcap_{k=1}^{\infty} B_k\right) \\
&= m\left(\bigcap_{N=1}^{\infty} \bigcap_{k=1}^{N} B_k\right)
\end{aligned}
$$

We know: $\cdots \supseteq \bigcap_{k=1}^{N} B_k \supseteq \bigcap_{k=1}^{N+1} B_k \supseteq \cdots$, so by the continuity of measure:

$$= \lim_{N\to\infty} m\left(\bigcap_{k=1}^{N} B_k\right)$$

Using the property of the B_k's we proved in part (a):

$$= \lim_{N\to\infty} \prod_{k=1}^{N} m(B_k)$$

By definition of the B_k's:

$$= \lim_{N\to\infty} \prod_{k=1}^{N} (1 - m(A_k))$$

Rewriting in terms of an exponential and using the fact that log is continuous:

$$
\begin{aligned}
&= \exp\left(\lim_{N\to\infty} \sum_{k=1}^{N} \log(1 - m(A_k))\right) \\
&= \exp\left(\lim_{N\to\infty} \sum_{k=1}^{N} \left(-\sum_{\ell=1}^{\infty} \frac{m(A_k)^{\ell}}{\ell}\right)\right)
\end{aligned}
$$

If we truncate at the first term of the inner series:

$$
\begin{aligned}
&\leq \exp\left(\lim_{N\to\infty} -\sum_{k=1}^{N} m(A_k)\right) \\
&= \exp\left(-\sum_{k=1}^{\infty} m(A_k)\right)
\end{aligned}
$$

And since $\sum_{k=1}^{\infty} m(A_k) = \infty$:

$$= 0$$

Measures cannot be less than 0: this shows that $m\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right) = 0$, which is equivalent to $m\left(\bigcup_{k=1}^{\infty} A_k\right) = 1$.

Problem 6

Let μ_s be Lebesgue measure on $S = [0, 1]$; let μ_t be the counting measure on S, i.e., $\mu_t(B) =$ the number of elements of B, for any finite $B \subset S$. Let D be the diagonal $D = \{(s, t) : s = t\}$ in $S \times S$, and f the characteristic function of D.

(a) Show that, for any s,

$$\int_S f(s,t) d\mu_t = 1$$

(b) Show that, for any t,

$$\int_S f(s,t) d\mu_s = 0$$

(c) Show that

$$\int_{S\times S} f(s,t) d\mu$$

where $\mu = \mu_s \otimes \mu_t$, does not exist (as a finite number).

Solution:

(a) For a fixed s, $f(s, t) = 0$ for all $t \neq s$. So as a function of t on S, we can write $f(s, t)$:

$$f(s,t) = \begin{cases} 0 & \text{if } t \neq s \\ 1 & \text{if } t = s \end{cases}$$

Evaluating the integral:

$$\begin{aligned} \int_S f(s,t) d\mu_t &= \int_{\{s\}} f(s,t) d\mu_t \\ &= \int_{\{s\}} 1 d\mu_t \\ &= \mu_t(\{s\}) \\ &= 1 \end{aligned}$$

(b) For a fixed t, we have a similar piecewise decomposition of $f(s, t)$:

$$f(s,t) = \begin{cases} 0 & \text{if } s \neq t \\ 1 & \text{if } s = t \end{cases}$$

Evaluating the integral:

$$\begin{aligned} \int_S f(s,t) d\mu_s &= \int_{\{t\}} f(s,t) d\mu_s \\ &= \int_{\{t\}} 1 d\mu_s \\ &= \mu_s(\{t\}) \\ &= 0 \end{aligned}$$

(c) Fubini's theorem doesn't hold because the counting measure is not σ-finite.
Looking at the integral and recalling the definition of measure:

$$\begin{aligned} \int_{S\times S} f(s,t) d\mu &= \int_{S\times S} \chi_D(s,t) d\mu \\ &= \mu(D) \\ &= \infty, \text{ because the measure of the diagonal in } [0,1] \text{ is infinite.} \end{aligned}$$

***$This doesn't quite work:$ because it ends up being $\infty \cdot 0$, but if we approximate by rectangles it'll be $\infty \cdot$(small finite), so it really does end up being infinite.

$\square$