Analysis Prelim January 2012

Sarah Arpin
University of Colorado Boulder
Mathematics Department
Sarah.Arpin@colorado.edu

Problem 1

(a) Let A be a measurable subset of $[0,1]$. Define the function $f : [0,1] \to \mathbb{R}$ by setting $f(x) = \mu(A \cap [0,x])$; here μ is Lebesgue measure. Show that f is absolutely continuous.

(b) Does there exist a measurable set $A \subset [0,1]$ such that one has

$$\mu(A \cap [a,b]) = \frac{1}{2}(b-a)$$

for every interval $[a,b] \subset [0,1]$?

Solution:

(a) f is absolutely continuous iff $\forall \epsilon > 0$, there exists $\delta > 0$ such that if

$$\sum_{k=1}^{N}(b_k - a_k) < \delta \text{ for some collection of disjoint subintervals } \{(a_k, b_k)\}_{k=1}^{N} \subseteq [0,1]$$

then:

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \epsilon$$

Using the definition of the function f:

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| = \sum_{k=1}^{N} |\mu(A \cap [0,b_k]) - \mu(A \cap [0,a_k])|$$

By the excision property:

$$= \sum_{k=1}^{N} \mu(A \cap [a_k, b_k])$$

Since $A \cap [a_k, b_k] \subseteq [a_k, b_k]$:

$$\leq \sum_{k=1}^{N} \mu([a_k, b_k])$$

$$= \sum_{k=1}^{N} |b_k - a_k|$$

So if we set $\delta = \epsilon$, we get the desired result. Thus, f is absolutely continuous.

(b) Not possible. Proof by contradiction.
Suppose there does exist $A \subset [0,1]$ such that $\mu(A \cap [a,b]) = \frac{1}{2}(b-a)$ for every interval $[a,b] \subset [0,1]$. Then:

$$\mu(A \cap [b,a]) = f(b) - f(a) = \frac{1}{2}(b-a)$$

Consider the interval $[0,x]$ where x is a variable:

$$f(x) - f(0) = \frac{1}{2}(x-0)$$

$$f'(x) = \frac{1}{2}$$

Thus $f'(x) = \frac{1}{2}$ a.e. Looking at the original definition of $f(x)$ and finding $f'(x)$, this would imply:

$$\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx} \int_{A \cap [0,x]} 1 dx \\
&= \frac{d}{dx} \int_0^x \chi_A(t) dt \\
&= \chi_A(x) (\text{by the fundamental theorem of calculus})
\end{aligned}$$

But $\chi_A(x) = 0$ or 1, so this is not possible.

$\square$

Problem 2

Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Set $f_n(x) = f(x + \frac{1}{n})$. Show that the sequence f_n converges to f in L^p. Is this true for $p = \infty$?

Solution:
The continuous functions of compact support $C_C(\mathbb{R})$ are dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.
First, suppose $f \in C_C(\mathbb{R})$. Then, there exists $[a, b] \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}} f d\mu = \int_a^b f d\mu$$

Since f is continuous, we know $f_n(x) \to f(x)$ pointwise. The function $|\cdot|^p$ is also continuous, so this means $|f(x) - f_n(x)|^p \to 0$ pointwise as $n \to \infty$.
Also, by the extreme value theorem there exists $M > 0$ such that $|f(x)| \leq M$ on $[a, b]$. We will show that $|f - f_n|^p$ is bounded as well, and then justify the switching of the limit with the integral by the Lebesgue Dominated Convergence Theorem.

$$\begin{aligned}
|f(x) - f_n(x)|^p &\leq (|f(x)| + |f_n(x)|)^p \cdot \chi_{[a,b]} \\
&\leq (2M)^p \cdot \chi_{[a,b]}
\end{aligned}$$

The function $(2M)^p \cdot \chi_{[a,b]}$ is integrable:

$$\int_{\mathbb{R}} (2M)^p \cdot \chi_{[a,b]} d\mu = \int_a^b (2M)^p = (b - a)(2M)^p < \infty$$

So the function $|f(x) - f_n(x)|^p$ is dominated by the integrable function $(2M)^p \cdot \chi_{[a,b]}$, and by the Lebesgue Dominated Convergence Theorem we are allowed to switch the limit and the integral:

$$\begin{aligned}
\lim_{n \to \infty} \int_{\mathbb{R}} |f(x) - f_n(x)|^p dx &= \int_{\mathbb{R}} 0 dx \\
&= 0
\end{aligned}$$

This shows that $\|f - f_n\|_p^p \to 0$ as $n \to \infty$, so by taking the p-th route we show $\|f - f_n\|_p \to 0$, as desired.
Now that we have shown that the result holds for continuous functions of compact support, we will use the fact that these functions are dense in $L^p(\mathbb{R})$ to show that $L^p(\mathbb{R})$ has the property as well.
Take $f \in L^p(\mathbb{R})$. There exists $g \in C_C(\mathbb{R})$ such that $\|f - g\|_p < \sqrt[p]{\epsilon/2}$.
Let $g_n(x) := g(x + \frac{1}{n}$.

Considering the integral in question, fix an arbitrarily small $\epsilon > 0$:

$$\begin{aligned}
\lim_{n\to\infty} \|f - f_n\|_p^p &= \int_{\mathbb{R}} |f(x) - f_n(x)|^p dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - g(x) + g(x) - g_n(x) + g_n(x) - f_n(x)|^p dx \\
&\leq \lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - g(x)|^p dx + \lim_{n\to\infty} \int_{\mathbb{R}} |g(x) - g_n(x)|^p dx + \lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - g_n(x)|^p dx \\
&\text{In the last integral, substitute } x + \frac{1}{n} = y\text{:} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - g(x)|^p dx + \lim_{n\to\infty} \int_{\mathbb{R}} |g(x) - g_n(x)|^p dx + \lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - g(x)|^p dx \\
&= \|f - g\|_p^p + \lim_{n\to\infty} \int_{\mathbb{R}} |g(x) - g_n(x)|^p dx + \|f - g\|_p^p \\
&< \epsilon + 0
\end{aligned}$$

Since ϵ can be chosen to be arbitrarily small, this shows $\|f - f_n\|_p^p \to 0$ as $n \to \infty$, and thus $\|f - f_n\|_p \to 0$ as desired.

□

Problem 3

Let f_n be a sequence of continuous functions on $[0, 1]$ such that $|f_n(x)| \leq 1$ for all $n \in \mathbb{N}, x \in [0, 1]$. Let K be a continuous function on $[0, 1] \times [0, 1]$. Define a sequence of functions g_n on $[0, 1]$ by

$$g_n(x) := \int_0^1 K(x, y) f_n(y) dy$$

Show that the sequence g_n contains a uniformly convergent subsequence.

Solution:
First, show that $g_n(x)$ is a continuous function for each n. Fix $\epsilon > 0$.

$$\begin{aligned}
|g_n(x_1) - g_n(x_2)| &= \left| \int_0^1 (K(x_1, y) - K(x_2, y)) f_n(y) dy \right| \\
&\leq \int_0^1 |K(x_1, y) - K(x_2, y)| \cdot |f_n(y)| dy \\
&\leq \int_0^1 |K(x_1, y) - K(x_2, y)| \cdot |f_n(y)| dy \\
&\text{Since } K \text{ is continuous on a compact set, } K \text{ must be uniformly continuous.} \\
&\text{Thus, there exists } \delta > 0 \text{ such that if } |x_1 - x_2| < \delta\text{, then } |K(x_1, y) - K(x_2, y)| < \epsilon : \\
&\text{Supposing we take such } x_1, x_2 \text{ with } |x_1 - x_2| < \delta\text{:} \\
&\leq \int_0^1 \epsilon \cdot 1 dy \\
&= \epsilon
\end{aligned}$$

So g_n is continuous. The continuity condition does not depend on n, so $\{g_n\}$ is equicontinuous.
By Arzela-Ascoli: A subset $\{g_n\} \subset C([0, 1])$ contains a uniformly convergent subsequence if and only if it is bounded and equicontinuous.
We have already shown that $\{g_n\}$ is equicontinuous. Next, show that it is bounded.

Note that, by the extreme value theorem, there exists M such that $|K(x,y)| \leq M$ for all $(x,y) \in [0,1] \times [0,1]$.

$$\begin{aligned}
|g_n(x)| &= \left| \int_0^1 K(x,y) f_n(y) dy \right| \\
&\leq \int_0^1 |K(x,y)| \cdot |f_n(y)| dy \\
&\leq \int_0^1 M \cdot 1 dy \\
&= M
\end{aligned}$$

So $|g_n(x)| \leq M$ for all n and for all $x \in [0,1]$.

$\square$

Problem 4

Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$, and suppose that for every $a > 0$ the infinite series $\sum_{n=1}^{\infty} \mu(\{x \in [0,1] : |f_n(x)| > a\})$ converges; here μ is Lebesgue measure. Prove that $\lim_{n\to\infty} f_n(x) = 0$ for almost every $x \in [0,1]$.

Solution: By the Borel-Cantelli Lemma, if the series $\sum_{n=1}^{\infty} \mu(\{x \in [0,1] : |f_n(x)| > a\})$ converges, then almost every $x \in [0,1]$ belongs to at most finitely many of the sets $\{x \in [0,1] : |f_n(x)| > a\}$.
Fix $\epsilon > 0$.
By hypothesis, the series $\sum_{n=1}^{\infty} \mu(\{x \in [0,1] : |f_n(x)| > \epsilon/2\})$ converges, so almost every $x \in [0,1]$ belongs to at most finitely many of the sets $\{x \in [0,1] : |f_n(x)| > \epsilon/2\}$.
Thus, there exists $N \in \mathbb{N}$ such that $\mu(\{x \in [0,1] : |f_n(x)| > \epsilon/2\}) = 0$ for all $n \geq N$. Thus, almost every $x \in [0,1]$ belongs to $\{x \in [0,1] : |f_N(x)| \leq \epsilon/2\}$, so for almost every $x \in [0,1]$, $|f_n(x)| \leq \epsilon/2 < \epsilon$ for all $n \geq N$. This shows $\lim_{n\to\infty} |f_n(x)| = 0$, since ϵ can be made arbitrarily small.

$\square$

Problem 5

Let $A \subset \mathbb{R}$ be a set of zero Lebesgue measure. Prove that it can be 'translated completely into the set of irrationals', that is, there exists a $c \in \mathbb{R}$ such that $A + c \subset \mathbb{R} \setminus \mathbb{Q}$, where $A + c := \{x + c : x \in A\}$.

Solution:
By contradiction, suppose that there exists $q \in \mathbb{Q}$ such that $q \in A + (-r)$ for every $r \in \mathbb{R}$.
Thus, for every $r \in \mathbb{R}$, there exists some q such that:

$$q = a + (-r) \text{ for some } a \in A$$

This implies $r = a - q$. Since this holds for every $r \in \mathbb{R}$:

$$\mathbb{R} \subseteq \bigcup_{q \in \mathbb{Q}} (A - q)$$

However, $m(A - q) = 0$, because Lebesgue measure is translation invariant.
Since the union over $\mathbb{Q}$ is countable, this would mean that $\mathbb{R}$ is a countable union of measure 0 sets, which would mean $\mathbb{R}$ is measure 0 itself (by the countable additivity of measure). Of course $\mathbb{R}$ is not measure 0, so this is not possible. Thus, there must exist some $r \in \mathbb{R}$ such that $A + r \subset \mathbb{R} \setminus \mathbb{Q}$.

$\square$

Problem 6

Let μ be the Lebesgue measure on the interval $[a, b]$. Let A_n, $n \geq 1$ be measurable subsets of $[a, b]$, and $f(x)$ the number of sets containing x, for $x \in [a, b]$. That is, $f(x) = \#(\{n \geq 1 : x \in A_n\})$
Prove that $f : [a, b] \to \mathbb{N} \cup \{+\infty\}$ is measurable and that

$$(b-a)\int_{\mathbb{R}} f^2(x)dx \geq \left[\sum_{k=1}^{\infty} \mu(A_k)\right]^2$$

Solution:

First, note that

$$f(x) = \sum_{n=1}^{\infty} \chi_{A_n}$$

So f is a nonnegative function on $[a, b]$.
By definition, f is measurable if and only if the sets $\{x \in [a, b] : f(x) \geq n\}$ are measurable for $n \in \mathbb{N}$.
Note how these sets are constructed:

$$\begin{aligned}
\{x \in [a,b] : f(x) \geq 0\} &= [a,b] \\
\{x \in [a,b] : f(x) \geq 1\} &= \bigcup_{n=1}^{\infty} A_n \\
\{x \in [a,b] : f(x) \geq 2\} &= \bigcup_{n_1 \neq n_2 \in \mathbb{N}} (A_{n_1} \cap A_{n_2}) \\
&\vdots \\
\{x \in [a,b] : f(x) \geq k\} &= \bigcup_{n_1, \ldots, n_k \in \mathbb{N} \text{ disjoint}} \left(\bigcap_{i=1}^{k} A_{n_i}\right)
\end{aligned}$$

Since the A_n are measurable, these are measurable sets as well, so f is a measurable function.
To show the specified inequality, consider two cases.
First, if $f \notin L^2(\mathbb{R})$, then:

$$\int_{\mathbb{R}} f^2(x)dx \geq \infty \geq \left[\sum_{k=1}^{\infty} \mu(A_k)\right]^2$$

So the desired result holds.
Next, suppose $f \in L^2(\mathbb{R})$. $\chi_{[a,b]} \in L^2$ as well, so by Cauchy-Schwarz:

$$\begin{aligned}
\int_{\mathbb{R}} f(x)dx &= \int_{\mathbb{R}} f(x) \cdot \chi_{[a,b]} dx \\
&\leq \left\|\chi_{[a,b]}\right\|_{L^2} \cdot \|f\|_{L^2} \\
&= (b-a)^{1/2} \left(\int_{\mathbb{R}} f^2(x)dx\right)^{1/2}
\end{aligned}$$

Squaring both sides:

$$\left(\int_{\mathbb{R}} f(x)dx\right)^2 \leq (b-a)\int_{\mathbb{R}} f^2(x)dx$$

So it remains only to show that $\left(\int_{\mathbb{R}} f(x)dx\right)^2 = \left[\sum_{k=1}^{\infty} \mu(A_k)\right]^2$.

$$\left(\int_{\mathbb{R}} f(x)dx\right)^2 = \left(\int_{\mathbb{R}} \sum_{k=1}^{\infty} \chi_{A_k}(x)dx\right)^2$$

By Tonelli's theorem, since the integrand is nonnegative we can switch the sum and integral:

$$= \left(\sum_{k=1}^{\infty} \int_{\mathbb{R}} \chi_{A_k}\right)^2$$

$$= \left(\sum_{k=1}^{\infty} \mu(A_k)\right)^2$$

Which shows the desired result.

□