Analysis Prelim August 2012

Sarah Arpin
University of Colorado Boulder
Mathematics Department
Sarah.Arpin@colorado.edu

Problem 1

Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of real-valued Lebesgue measurable functions defined on $[0, 1]$. Suppose $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in [0, 1]$.

(a) Is f necessarily Lebesgue measurable? If yes, prove it, and if no, provide a counterexample.

(b) Give a condition on $\{f_n : n \in \mathbb{N}\}$ that guarantees

$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$$

Be sure to prove that your condition implies the desired conclusion.

(c) Give an example of a sequence of Lebesgue measurable functions defined on $[0, 1]$ that violates your condition in (b) and such that

$$\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$$

Solution:

(a) Yes. f is measurable iff, for every $c > 0$, the set $\{x : f(x) > c\}$ is measurable.

$$\{x : f(x) > c\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > c\}$$

The countable intersection of measurable sets is measurable, so $\{x : f(x) > c\}$ is measurable, as desired. Thus, f is measurable.

(b) If $f_1 \leq f_2 \leq f_3 \leq \cdots$, then the Monotone Convergence Theorem applies and we may switch the limit and the integral.
If $|f_n(x)| \leq g(x)$ for all n and for all x, for some g which is integrable, then the Lebesgue Dominated Convergence Theorem applies and we may switch the limit and the integral.

(c) consider the sequence of functions $\{f_n\}$ defined:

$$f_n(x) = n \cdot \chi_{[0,1/n]}$$

$f_n(x) \to 0$ pointwise a.e. on $[0, 1]$ (just not at 0).
However,

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \int_0^{\frac{1}{n}} ndx = \lim_{n\to\infty} 1 = 1$$

So we do *not* have $\lim\limits_{n\to\infty} \int_0^1 f_n = \int_0^1 f$.

$\square$

Problem 2

Let $f \in L^1(\mathbb{R})$, the set of Lebesgue integrable functions over $\mathbb{R}$. Prove that

$$\lim_{x\to 0}\int_{\mathbb{R}} |f(x+t)-f(t)|dt = 0$$

You may use the fact that the space $C_C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with compact support is dense in $L^1(\mathbb{R})$, with respect to the L^1 norm.

Solution:
First, consider a function $g \in C_C(\mathbb{R})$. By definition of compact support, there exists $[a,b] \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}} |g(x+t)-g(t)|dt = \int_a^b |g(x+t)-g(t)|dt$$

Note that $\lim_{x\to 0} |g(x+t)-g(t)| = 0$. We will show that it is possible to pass the limit through the integral using the Lebesgue Dominated Convergence Theorem:
Since $[a,b]$ is closed and bounded and g is continuous, the extreme value theorem applies and gives us a finite $M > 0$ such that $|g(x)| \leq M$ on $[a,b]$.
Since $\int_a^b 2M dt = (b-a)\cdot 2M < \infty$, $2M$ is an integrable function. Also, $|g(x+t)-g(t)| \leq 2M$ by the triangle inequality. This allows us to apply the Lebesgue Dominated Convergence Theorem to pass the limit through the integral:

$$\lim_{x\to 0}\int_{\mathbb{R}} |g(x+t)-g(t)|dt = \lim_{x\to 0}\int_a^b |g(x+t)-g(t)|dt = \int_a^b \lim_{x\to 0}|g(x+t)-g(t)|dt = 0$$

The statement holds for continuous functions of compact support.
Fix $\epsilon > 0$ and consider $f \in L^1(\mathbb{R})$.
Since $C_C(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, there exists $g \in C_C(\mathbb{R})$ such that $\|f-g\|_{L^1} < \epsilon/2$.

$$\begin{aligned}
\int_{\mathbb{R}} |f(x+t)-f(t)|dt &= \int_{\mathbb{R}} |f(x+t)-g(x+t)+g(x+t)-g(t)+g(t)-f(t)|dt \\
&\leq \int_{\mathbb{R}} |f(x+t)-g(x+t)|dt + \int_{\mathbb{R}} |g(x+t)-g(t)|dt + \int_{\mathbb{R}} |g(t)-f(t)|dt \\
&< \epsilon/2 + \int_{\mathbb{R}} |g(x+t)-g(t)|dt + \epsilon/2 \\
\lim_{x\to 0}\int_{\mathbb{R}} |f(x+t)-f(t)|dt &\leq \lim_{x\to 0}(\epsilon + \int_{\mathbb{R}} |g(x+t)-g(t)|dt) \\
&= \epsilon
\end{aligned}$$

Since ϵ can be made arbitrarily small, we have our result.

$\square$

Problem 3

Let f be a measurable function on $\mathbb{R}$ with $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

(a) Prove that for all $p \in (1,\infty)$, $f \in L^p(\mathbb{R})$.

(b) Prove that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

Solution:

(a) Take $p \in (1, \infty)$. Since $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)| < \infty$, we know $\left\||f|^{p-1}\right\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|^{p-1} < \infty$.
Thus, $|f|^{p-1} \in L^\infty(\mathbb{R})$.
Since $|f| \in L^1(\mathbb{R})$ and $|f|^{p-1} \in L^\infty(\mathbb{R})$, Hölder's inequality gives us $|f|^p \in L^1(\mathbb{R})$ and:

$$\int_{\mathbb{R}} |f(x)|^p dx \leq \|f\|_{L^1} \cdot \left\||f|^{p-1}\right\|_\infty < \infty$$

Which shows $\|f\|_p^p < \infty$, so $f \in L^p(\mathbb{R})$.

(b) Note that $\mathbb{R}$ is σ-finite, so there exist sets A_n of finite measure such that $A_n \nearrow \mathbb{R}$ and $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$. For a small $\epsilon > 0$, define the set:

$$S := \{x \in \mathbb{R} : |f(x)| \leq \|f\|_\infty - \epsilon\}$$

By construction, $\mu(S) > 0$. Also:

$$\begin{aligned} S &= S \cap \mathbb{R} \\ &= S \cap \left(\bigcup_{n=1}^{\infty} A_n\right) \\ &= \bigcup_{n=1}^{\infty} (S \cap A_n) \end{aligned}$$

Since $\mu(S) > 0$, there exists some k such that $\mu(S \cap A_k) > 0$, by countable subadditivity of measure. Also, $\mu(A_k) < \infty$, so $\mu(S \cap A_k) \in (0, \infty)$.
Moving forward with this in mind, for any p:

$$\begin{aligned} \|f\|_p &= \left(\int_{\mathbb{R}} |f|^p d\mu\right)^{1/p} \\ &\text{Since } |f|^p \geq 0 \text{ and } S \subseteq \mathbb{R}: \\ &\geq \left(\int_{S\cap A_k} |f|^p d\mu\right)^{1/p} \\ &\text{Since } |f| \geq \|f\|_\infty - \epsilon \text{ on } S: \\ &\geq \left(\int_{S\cap A_k} |\, \|f\|_\infty - \epsilon|^p\right)^{1/p} \\ &= (\mu(S \cap A_k))^{1/p} |\, \|f\|_\infty - \epsilon| \end{aligned}$$

Since $\mu(S \cap A_k) \in (0, \infty)$, as $p \to \infty$, $\mu(S \cap A_k)^{1/p} \to 1$. Thus:

$$\liminf_{p\to\infty} \|f\|_p \geq |\, \|f\|_\infty - \epsilon|$$

Since this holds for all $\epsilon > 0$:

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty$$

To get the reverse inequality, recall from part (a) that $|f|^{p-1} \in L^\infty(\mathbb{R})$. Then use Hölder's inequality:

$$\begin{aligned} \|f\|_p &= \left(\int_{\mathbb{R}} |f|^p d\mu\right)^{1/p} \\ &\leq \left(\||f|\|_{L^1} \left\||f|^{p-1}\right\|_\infty\right)^{1/p} \\ &= \|f\|_{L^1}^{1/p} \|f\|_\infty^{\frac{p-1}{p}} \end{aligned}$$

Since $\|f\|_{L^1} < \infty$, as $p \to \infty$, $\|f\|_{L^1}^{1/p} \to 1$ and $\|f\|_\infty^{\frac{p-1}{p}} \to \|f\|_\infty$. Thus:

$$\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty$$

Since we have $\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty \le \liminf_{p\to\infty} \|f\|_p$, equality must hold:

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

□

Problem 4

A function $f : [a,b] \to \mathbb{R}$ is said to be Lipschitz on $[a,b]$ provided there is a constant $M > 0$ such that $|f(x) - f(y)| \le M|x-y|$ for all $x, y \in [a,b]$.

(a) Prove that if $g : [a,b] \to [c,d]$ is absolutely continuous on $[a,b]$, and $f : [c,d] \to \mathbb{R}$ is Lipschitz on $[c,d]$, then $f \circ g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$.

(b) By using part (a) or otherwise, prove that any Lipschitz function f defined on $[a,b]$ is absolutely continuous. Is the converse true, i.e. is an absolutely continuous function $f : [a,b] \to \mathbb{R}$ necessarily Lipschitz? Either prove this is true, or provide a counterexample.

Solution:
Note:

Cont. Differentiable $\subseteq$ Lipschitz $\subseteq$ abs. continuous $\subseteq$ bounded variation $\subseteq$ diff. a.e.

(a) Fix $\epsilon > 0$. We want to show that there exists a $\delta > 0$ such that if $\{(a_k, b_k)\}$ is a collection of pairwise disjoint subintervals of $[a,b]$ such that

$$\sum_{k=1}^{\infty} (b_k - a_k) < \delta$$

implies

$$\sum_{k=1}^{\infty} |(f \circ g)(b_k) - (f \circ g)(a_k)| < \epsilon$$

Let M be the Lipschitz constant corresponding to f, as described above.
Since g is absolutely continuous, there exists $\delta > 0$ such that if

$$\sum_{k=1}^{\infty} (b_k - a_k) < \delta$$

then

$$\sum_{k=1}^{\infty} |g(b_k) - g(a_k)| < \frac{\epsilon}{M}$$

Consider now the desired sum, with respect to a partition of $[a,b]$ satisfying the condition imposed by the δ selected above:

$$\begin{aligned}
\sum_{k=1} |f(g(b_k)) - f(g(a_k))| &\le \sum_{k=1} M|g(b_k) - g(a_k)|, \text{ since } f \text{ is Lipschitz.} \\
&= M \sum_{k=1} |g(b_k) - g(a_k)| \\
&< M(\epsilon/M), \text{ by selection of the } (a_k, b_k) \text{ subintervals.} \\
&= \epsilon
\end{aligned}$$

Which shows that $f \circ g$ is absolutely continuous.

(b) Every Lipschitz function is absolutely continuous. Let f be Lipschitz with Lipschitz constant $M \in (0, \infty)$. Fix $\epsilon > 0$. Pick a partition $\{(x_k, y_k)\}_{k=1}^N$ such that $\sum_{k=1}^N |y_k - x_k| < \epsilon/M$. Then:

$$\begin{aligned}\sum_{k=1}^N |f(y_k) - f(x_k)| &\leq \sum_{k=1}^N M|y_k - x_k|, \text{ since } f \text{ is Lipschitz.} \\ &= M \sum_{k=1} |y_k - x_k| \\ &< M(\epsilon/M) \\ &= \epsilon\end{aligned}$$

Absolutely continuous functions need not be Lipschitz. Consider the function $f(x) = \sqrt{x}$ on $(0, 1]$. It is absolutely continuous, but it is not Lipschitz.

- Not Lipschitz: $f(x)$ is continuously differentiable on its domain $(0, 1]$. A differentiable function is Lipschitz if and only if it has bounded first derivative, and the derivative of $f(x) \to \infty$ as $x \to 0$;

$$f'(x) = \frac{1}{2\sqrt{x}} \to \infty \text{ as } x \to 0$$

  So $f(x)$ is not Lipschitz.

- Absolutely Continuous:
  A function $f(x)$ defined on $(0, 1]$ is absolutely continuous if and only if there exists a Lebesgue integrable function g such that:

$$f(x) = f(0) + \int_0^x g(t)dt$$

  $f(x) = \sqrt{x}$ does have such an integrable derivative, namely $f'(x) = \frac{1}{2\sqrt{x}}$

□

Problem 5

Let $f \in L^1[-\pi, \pi]$, and for $n \in \mathbb{Z}$, define $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt$, where $e^{i\theta} = \cos\theta + i \sin\theta$.

(a) Prove that $\lim_{|n|\to\infty} c_n$ exists.

(b) Is the limit in (a) independent of f? If so, prove it. If not, give examples of $f_1, f_2 \in L^1[-\pi, \pi]$ with different limits arising in (a).

Solution:
This question is precisely the Riemann Lebesgue Lemma.

(a) First, consider the case where $f(x) \in C_C^\infty([-\pi, \pi])$ (a smooth function of compact support). Let

$\text{supp}(f) \subseteq [a,b] \subset [-\pi, \pi]$.

$$\begin{aligned}
\lim_{|n|\to\infty} |c_n| &= \lim_{|n|\to\infty} \left| \frac{1}{2\pi} \int_{\mathbb{R}} f(t)e^{-int}dt \right| \\
&= \lim_{|n|\to\infty} \frac{1}{2\pi} \left| \int_a^b f(t)e^{-int}dt \right| \\
&\text{Using integration by parts:} \\
&= \lim_{|n|\to\infty} \frac{1}{2\pi} \left| \left(\frac{f(t)e^{-int}}{-in}\Big|_a^b + \frac{1}{in}\int_a^b f'(t)e^{-int}dt \right) \right| \\
&\leq \lim_{|n|\to\infty} \left(\frac{1}{2\pi in} \int_a^b |f'(t)|dt \right) \\
&= 0 \text{ Since } \int_a^b |f'(t)|dt < \infty.
\end{aligned}$$

The smooth functions of compact support are dense in L^1 with respect to the L^1 norm. Fix $\epsilon > 0$. There exists $g \in C_C^\infty([-\pi, \pi])$ such that

$$\|g - f\|_{L^1} < \epsilon$$

Consider the limit in question:

$$\begin{aligned}
\lim_{|n|\to\infty} |c_n| &= \lim_{|n|\to\infty} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt \right| \\
&= \lim_{|n|\to\infty} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t) - g(t) + g(t))e^{-int}dt \right| \\
&\leq \lim_{|n|\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - g(t)| \cdot |e^{-int}|dt + \lim_{|n|\to\infty} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{-int}dt \right| \\
&\leq \lim_{|n|\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - g(t)|dt + 0 \\
&= \lim_{|n|\to\infty} \frac{1}{2\pi} \|f - g\|_{L^1} \\
&< \frac{\epsilon}{2\pi}
\end{aligned}$$

*Alternatively, we could use the simple functions, which are dense in L^p for $p \in [1, \infty]$.

(b) The limit is 0, which does not depend on the choice of f.

□

Problem 6

Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Prove that there exists $x \in [0, \frac{3}{4}]$ with $f(x) = f(x + \frac{1}{4})$.

Solution:
Proof by contradiction. Suppose no such x exists. Then, $f(x + \frac{1}{4}) - f(x) \neq 0$ for any $x \in [0, \frac{3}{4}]$.
This is a continuous function, so the intermediate value theorem applies and we see $f(x + \frac{1}{4}) - f(x) > 0$ every on $[0, \frac{3}{4}]$ or $f(x + \frac{1}{4}) - f(x) < 0$ everywhere on $[0, \frac{3}{4}]$.
Without loss of generality (we can always multiply by -1), suppose $f(x+\frac{1}{4}) - f(x) > 0$ everywhere on $[0, \frac{3}{4}]$.
By plugging in $x = 0, \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$:

$$f(0) > f(\frac{1}{4}) > f(\frac{1}{2}) > f(\frac{3}{4}) > f(1)$$

But this contradicts $f(0) = f(1)$, so our hypothesis must be false.

$\square$